

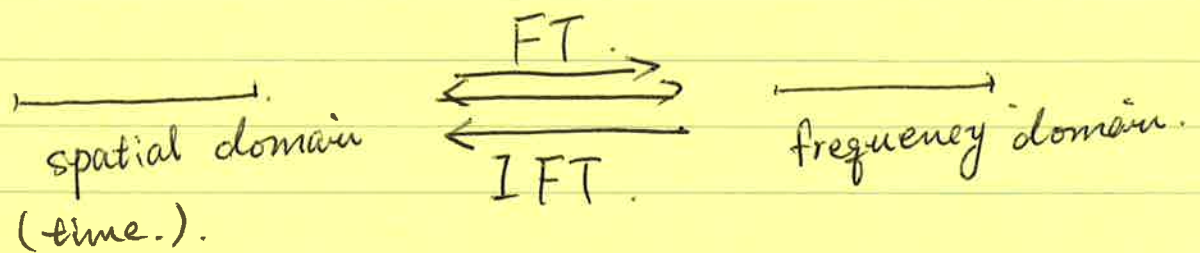
(P1)

Fourier Transform / Fast FT. Ryan.

Fourier transform.

f integrable function

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} e^{-2\pi i \xi x} f(x) dx.$$



IFT.

$$f(x) = \int_{-\infty}^{+\infty} e^{2\pi i \xi x} \hat{f}(\xi) d\xi.$$

Each value of $\hat{f}(\xi)$ corresponds

to a wave $\hat{f}(\xi) \cdot e^{2\pi i \xi x}$.

amplitude. frequency.

(P₃)

$$\Delta F_{11} \quad 0 \leq j', k' < \frac{N}{2}$$

$$(\overline{F}_{11})_{j', k'} = e^{-\frac{2\pi i}{N} \cdot (j' \cdot 2k')} = e^{-2\pi i \frac{j'k'}{(\frac{N}{2})}}$$

$$\text{So } F_{11} = F_{\frac{N}{2}}$$

$$\Delta F_{21} \quad 0 \leq j', k' < \frac{N}{2}$$

$$(\overline{F}_{21})_{\substack{j+\frac{N}{2} \\ j+\frac{N}{2}}, k'} = e^{-\frac{2\pi i}{N} (j+\frac{N}{2}) \cdot 2k'}$$

$$= e^{-\frac{2\pi i}{N} j \cdot 2k'} e^{-\frac{2\pi i}{N} \cdot \frac{N}{2} \cdot 2k'}$$

$$= e^{-\frac{2\pi i j' k'}{(\frac{N}{2})}}$$

$$\text{So } F_{21} = F_{\frac{N}{2}}$$

(P₄)

$$\Delta \quad F_{12} \quad 0 \leq j', k' < \frac{N}{2}$$

$$(F_{12})_{j', 2k'+1} = e^{-\frac{2\pi i}{N} (j' \cdot (2k'+1))}$$

$$= e^{-\frac{2\pi i j' \cdot 2k'}{N}} \cdot e^{-\frac{2\pi i}{N} j'}$$

$$= e^{-\frac{2\pi i j'}{N}} \cdot e^{-\frac{2\pi i j' k'}{\frac{N}{2}}}$$

$$F_{12} = \begin{pmatrix} e^{-\frac{2\pi i}{N} \cdot 0} & & & \\ & e^{-\frac{2\pi i}{N} \cdot 1} & & \\ & & \dots & \\ & & & e^{-\frac{2\pi i}{N} \cdot \frac{N}{2}} \end{pmatrix}$$

$$F_{\frac{N}{2}}$$

$$= \Omega_{\frac{N}{2}} F_{\frac{N}{2}}$$

(P5).

$$\Delta \quad F_{22} \quad 0 \leq j', k' < \frac{N}{2}$$

$$\left(F_{22} \right)_{j'+\frac{N}{2}, 2k'+1}$$

$$= e^{-\frac{2\pi i}{N} (j'+\frac{N}{2})(2k'+1)}$$

$$= e$$

$$= e^{-\frac{2\pi i}{N} j' \cdot k'} \cdot \underbrace{e^{-2\pi i k'}}_1 \cdot e^{-\frac{2\pi i}{N} j'} \cdot \underbrace{e^{-2\pi i \cdot \frac{1}{2}}}_{-1}$$

$$= - e^{-\frac{2\pi i}{N} j'} \cdot e^{-2\pi i j' k' / (\frac{N}{2})}$$

$$F_{22} = - \Omega_{N/2} F_{N/2}$$

$$\Delta \quad F_N = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} F_{N/2} & \Omega_{N/2} F_{N/2} \\ \bar{F}_{N/2} & -\Omega_{N/2} F_{N/2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \Omega_{N/2} \end{pmatrix} \begin{pmatrix} F_{N/2} \\ F_{N/2} \end{pmatrix}$$

(P6).

$$\hat{X} = F_N \begin{pmatrix} X_e \\ X_o \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \Omega_{N/2} \end{pmatrix} \begin{pmatrix} F_{N/2} & \\ & F_{N/2} \end{pmatrix} \begin{pmatrix} X_e \\ X_o \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \\ & \Omega_{N/2} \end{pmatrix} \begin{pmatrix} F_{N/2} X_e \\ F_{N/2} X_o \end{pmatrix}$$

1 size n FFT \Rightarrow 2 size $N/2$ FFT.

★ Algorithm

△ func $y = \text{FFT}(x, n)$. — $T(n)$.

if ($n=1$)

$$y = x.$$

— $O(1)$.

else.

$$\omega_n = e^{-2\pi i/n}.$$

— $O(1)$

$$X_e = \begin{bmatrix} x_0 \\ x_2 \\ \vdots \end{bmatrix} \quad X_o = \begin{bmatrix} x_1 \\ x_3 \\ \vdots \end{bmatrix}$$

— " $O(n)$ "

diag(Ω_n) \rightarrow

$$d = [1, \omega_n, \omega_n^2, \dots]$$

— $O(n)$

$$y_T = \text{FFT}(X_e, n/2)$$

— $T(n/2)$

$$y_B = \text{FFT}(X_o, n/2)$$

— $T(n/2)$

$$y_B = d * y_B$$

— $O(n)$

$$y = \begin{bmatrix} y_T + y_B \\ y_T - y_B \end{bmatrix}$$

— $O(n)$

end.

(P7)

Complexity:

$$T(n) = 2T\left(\frac{n}{2}\right) + 3n.$$

$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + 3\frac{n}{2}\right) + 3n.$$

$$= 4T\left(\frac{n}{4}\right) + 2 \cdot 3n$$

$$= \dots$$

$$= 2^{\log n} T\left(\frac{n}{n}\right) + 3n \log n$$

$$\approx 3n \log n = O(n \log n).$$

Applications.

Circulant Matrix.

$$C = \begin{bmatrix} c_0 & c_{n-1} & \dots & c_2 & c_1 \\ c_1 & c_0 & & & c_2 \\ c_2 & c_1 & & & \vdots \\ \vdots & \vdots & & & \vdots \\ c_{n-1} & c_{n-2} & & & c_0 \end{bmatrix}$$

$$Cx = b \Leftrightarrow c * x = b.$$

$$c = (c_0 \ c_1 \ \dots \ c_{n-1})^T.$$

(P8)

$$b_k = \sum_{i=0}^{n-1} x_i c_{k-i}$$

$$\mathcal{F}(c * x) = \mathcal{F}c \cdot \mathcal{F}x.$$

$$\widehat{(c * x)} = b \quad O(N \log N)$$

$$\widehat{(b / c)} = x \quad O(N \log N)$$

Toeplitz Matrix.

$$A = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-n+1} \\ a_1 & \dots & \dots & \dots \\ \vdots & \dots & \dots & \dots \\ a_{n-1} & \dots & a_{-1} & a_0 \end{bmatrix}$$

$$C = \begin{bmatrix} a_0 & a_{-1} & \dots & a_{-n+1} & a_{n-1} & \dots & a_1 \\ a_1 & a_0 & a_{-1} & \dots & a_{-n+1} & a_{n-1} & \dots & a_2 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & \dots & \dots & a_0 & a_{-1} & \dots & a_{-n+1} \\ a_{-n+1} & \dots & \dots & \dots & a_0 & \dots & \dots \\ \vdots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_0 \end{bmatrix}$$

(P9) .

$$Ax = b$$

\Leftrightarrow

$$C \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} b \\ * \end{bmatrix} .$$

✓

~~can be~~

FFT

