P1

ICME Refresher Course. Ryan.

Differential Equations.

$$\begin{cases} \text{ODE} & \text{Ordinary Differential Equations.} \\ \text{PDE} & \text{Partial Differential Equations} \end{cases}$$

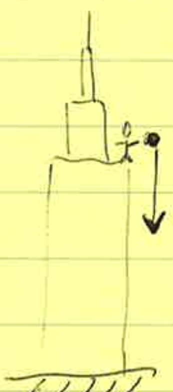
Notations: the derivative of y w.r.t x, t.

1st: $\dfrac{dy}{dx}$, y', $\dot{y}$, $\dfrac{\partial y}{\partial t}$, y_x.

2nd: $\dfrac{d^2y}{dx^2}$, y'', $\ddot{y}$, $\dfrac{\partial^2 y}{\partial t^2}$, y_{xx}.

nth: $\dfrac{d^ny}{dx^n}$, $y^{(n)}$, X, $\dfrac{\partial^n y}{\partial t^n}$, X.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(x)}{h}.$$

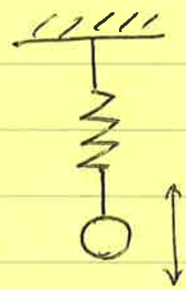


Falling Stone.

$y'' = g$ ← gravitational constant. ~~constant~~.

mass of the ball.

$$\underset{\uparrow \text{ mass of the ball}}{m} y'' + \underset{\downarrow \text{ spring constant}}{k} y = 0.$$



Vibrating mass on a spring

(P2)

First Order ODE

$F(x, y, y') = 0$ or $y' = f(x, y)$

(explicit form)

Usually has a general solution.

Direction Field.

$y' = y + x$

General solution involves a constant C.

$y' = 0 \Rightarrow y = C$.

In applications, we have to find a unique solution by determining a value of C from an initial condition. $y(x_0) = y_0$.

$$\begin{cases} F(x, y, y') = 0. \\ y(x_0) = y_0 \end{cases}$$ — IVP initial value problem

The solution is a particular solution.

P3

△ Separable ODE. (can't be used for higher order).

$$g(y)\,dy = f(x)\,dx.$$

$$\int g(y)\,dy = \int f(x)\,dx + C.$$

Ex: $y' = (x+1)e^{-x}y^2.$

solu: $y = \dfrac{1}{(x+2)e^{-x} - C}$

△ Exact ODE.

$$M(x,y)\,dx + N(x,y)\,dy = 0.$$

where $M\,dx + N\,dy$ is the differential

$$du = u_x\,dx + u_y\,dy.$$

Integrating factor: $F(x,y)$

$$F(x,y)\,M(x,y)\,dx + F(x,y)\,N(x,y)\,dy = 0.$$

is the differential $\underline{\underline{du}} = 0.$

Ex: $y = xy'$ $\qquad \dfrac{-y\,dx + x\,dy}{x^2} = -\dfrac{y}{x^2}\,dx + \dfrac{1}{x}\,dy$

$$= d\left(\frac{y}{x}\right) = 0.$$

P4

△ Linear ODE.

$$y' + p(x) y = r(x).$$

There is general solution.

Homogeneous Linear ODE.

$$y' + p(x) y = 0.$$

$$\frac{dy}{y} = -p(x)\, dx.$$

$$\ln|y| = \int -p(x)\, dx + C.$$

$$y = C e^{-\int p(x)\, dx}.$$

Nonhomogeneous Linear ODE.

$$y' + p(x) y = r(x).$$

Find $F(x)$ such that:

$$\frac{d\, Fy}{dx} = Fy' + Fpy.$$

$$Fp = F' \Rightarrow F = e^{\int p\, dx} := e^{h}.$$

P5.

$$(e^h y)' = r \cdot e^h.$$

$$e^h y = \int r e^h \, dx + C.$$

$$y = e^{-h}\left(\int r e^h \, dx + C\right)$$

$$= \underbrace{e^{-h}\int r e^h \, dx}_{\text{response to Input } r} + \underbrace{C e^{-h}}_{\text{response to initial value}}.$$

(P6).

Second-order Linear ODE.

$$y'' + p(x)\,y' + q(x)\,y = r(x).$$

Linear in y.

$$\begin{cases} \text{homogeneous} & y'' + p(x)y' + q(x)y = 0. \\ \text{nonhomogeneous} & y'' + p(x)y' + q(x)y = \underbrace{r(x)}_{\neq 0} \end{cases}$$

Superposition principle

Any soln of nonhomogeneous

$$y = \underbrace{y_p}_{\substack{\text{particular solution}\\ \text{of nonhomo}}} + \underbrace{y_h}_{\substack{\text{general solution}\\ \text{of homogeneous.}}}$$

△ homogeneous 2nd order ODE. with constant coefficients

$$y'' + ay' + by = 0.$$

Step 1 solve.

$$\lambda^2 + a\lambda + b = 0. \quad \text{for } \lambda,$$

P7

Step 2.

Case	Type of Roots	General Soln
I	real $\lambda_1 \neq \lambda_2$.	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
II	Double $\lambda_1 = \lambda_2$.	$y = (C_1 + C_2 x) e^{\lambda x}$
III	Complex. $\lambda_1 \neq \lambda_2$.	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

Type III: $\lambda_0 = -\frac{1}{2}a \pm i\omega$.

$$y = C_1 e^{-\frac{1}{2}ax + i\omega x} + C_2 e^{-\frac{1}{2}ax - i\omega x}.$$

$$= e^{-\frac{1}{2}ax} \left(C_1 e^{i\omega x} + C_2 e^{-i\omega x} \right)$$

$$= e^{-\frac{1}{2}ax} \left(C_1 (\cos \omega x + i \sin \omega x) + C_2 (\cos \omega x - i \sin \omega x) \right).$$

$$= e^{-\frac{1}{2}ax} \left(C_1 \cos \omega x + C_2 \sin \omega x \right)$$

(P8).

△. Euler - Cauchy Equations.

$$x^2 y'' + axy' + by = 0.$$

Step 1.

Assume $y = x^m$.

Step 2.

$$\cancel{x^2}\, m(m-1) \cancel{x} x^m + a m x^m + b x^m = 0.$$

$$m^2 + (a-1) m + b = 0.$$

Step 3.

Case	Type of Roots.	Soln.
I	$m_1 \neq m_2$ real	$y = c_1 x^{m_1} + c_2 x^{m_2}$.
II.	$m_1 = m_2$.	$y = (c_1 + c_2 \ln x) x^m$
III	$m_1 \neq m_2$ complex	$y = c_1 x^{m_1} + c_2 x^{m_2}$.

(P9).

△ Nonhomogeneous 2nd-order

Linear ODE. with constant coefficients.

$$y'' + ay' + by = r(x)$$

Term in $r(x)$	Choice for y_p
ke^{rx}	Ce^{rx}
kx^n	$K_n x^n + K_{n-1} x^{n-1} + \cdots + K_1 x + K_0$
$k\cos\omega x$, $k\sin\omega x$	$K\cos\omega x + M\sin\omega x$
$ke^{ax}\cos\omega x$, $ke^{ax}\sin\omega x$	$e^{ax}(K\cos\omega x + M\sin\omega x)$

Ex: $\quad y'' + 5y' + 4y = 10e^{-3x}$.

P10

Higher order Linear ODE.

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \cdots + p_1(x)\, y' + p_0(x)\, y = r(x).$$

Homogeneous. with constant coefficient.

$$y^{(n)} + a_{n-1}\, y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

Step 1 solve

$$\lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0 = 0.$$

Step 2.

For distinct roots.

$$y = C_1 e^{\lambda_1 x} + \cdots$$

For Double, triple, ... roots.

Mutiple roots.

$$y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + C_3 x^2 e^{\lambda_1 x} + \cdots$$

For double complex roots.

$$y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}.$$

(P11).

Uniqueness of the Solution.

Solve $y'' + p(x) y' + q(x) y = 0$

Existence.

Complicated!