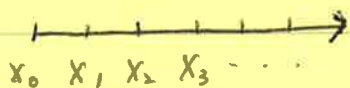


(P1)

Numerical Method for ODE

△ Euler's method.

$$y' = f(x, y)$$



$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_n = x_0 + nh$$

where h is the step size.

Taylor expansion of $y(x+h)$ at x .

$$y(x+h) = y(x) + hy'(x) + \frac{h^2}{2} y''(x) + \dots$$

For small h , $h^2, h^3 \ll h$.

$$\Rightarrow y(x+h) \approx y(x) + hy'(x)$$

The numerical scheme (Euler method)

$$y_{n+1} = y_n + h f(x_n, y_n)$$

(Forward Euler method.
Euler-Cauchy method)

(P2)

Error of the Euler method.

~~Taylor~~ Taylor series. $\exists \xi \in (x, x+h)$

$$y(x+h) = y(x) + h y'(x) + \underbrace{\frac{1}{2} h^2 y''(\xi)}$$

truncation error.
global error $O(h)$.

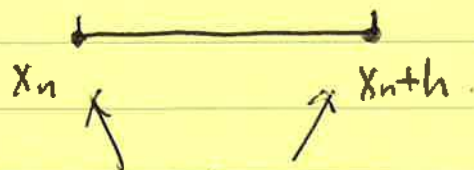
△ Improved Euler method.

$$y_{n+1}^* = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2} h (f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*))$$

truncation error $O(h^3)$

global error $O(h^2)$.



combine the Taylor expansions at both points.

(P3)

△ "Further improved Euler method"

Runge - Kutta Method.

Algorithm.

$$k_1 = h f(x_n, y_n).$$

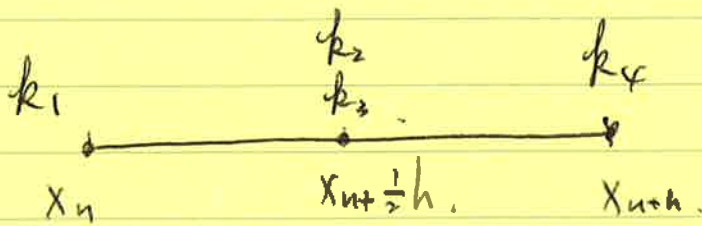
$$k_2 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1).$$

$$k_3 = h f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2).$$

$$k_4 = h f(x_n + h, y_n + k_3).$$

$$x_{n+1} = x_n + h.$$

$$y_{n+1} = y_n + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4).$$



Truncation error $O(h^5)$
global error $O(h^4)$.

You can design your own
higher order numerical ~~sch~~ method!

(P4)

△ Backward Euler Method.

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1}).$$

More stable, larger ~~converge~~ range of convergence.

△ ~~Central~~ diff.

Finite difference method.

$$y' = f(x, y).$$

$$\frac{y_{n+1} - y_n}{h} = f(x_n, y_n).$$

$$\frac{y_{n+1} - y_n}{h} = f(x_{n+1}, y_{n+1}).$$

$$? \frac{y_{n+1} - y_{n-1}}{2h} = f(x_n, y_n). \quad (\text{PDE}).$$

(P5)

y'' .

$$\frac{\frac{y_{n+1} - y_n}{h} - \frac{y_n - y_{n-1}}{h}}{h}$$