

P1

ICME Refresher Course. Ryan.

Differential Equations.

$\left\{ \begin{array}{l} \text{ODE} \quad \text{Ordinary Differential Equations.} \\ \text{PDE} \quad \text{Partial Differential Equations} \end{array} \right.$

Notations: the derivative of y w.r.t x, t .

1st: $\frac{dy}{dx}, y', \dot{y}, \frac{\partial y}{\partial t}, y_x$

2nd: $\frac{d^2y}{dx^2}, y'', \ddot{y}, \frac{\partial^2 y}{\partial t^2}, y_{xx}$

n-th: $\frac{d^n y}{dx^n}, y^{(n)}, X, \frac{\partial^n y}{\partial t^n}, X$

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{y(x+h) - y(x)}{h}$$



Falling Stone.

$y'' = g$ \leftarrow gravitational constant.



mass of the ball.

$m y'' + k y = 0$

spring constant

Vibrating mass on a spring

(P₂)

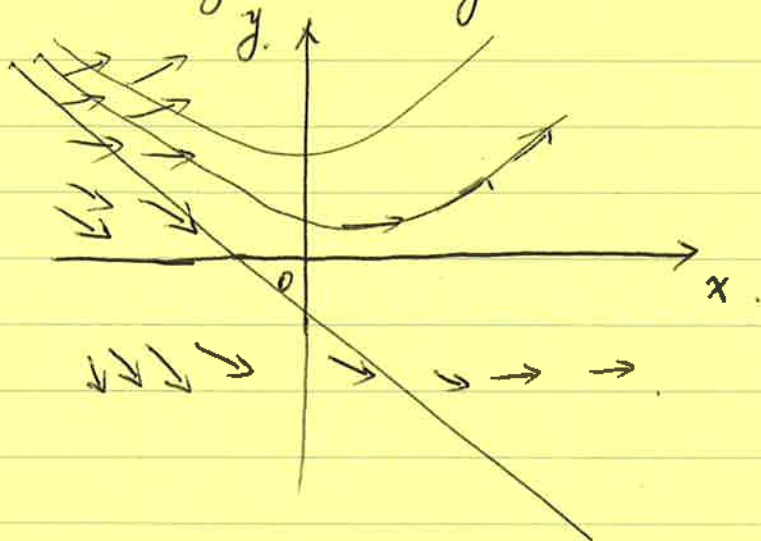
First Order ODE

$$F(x, y, y') = 0 \quad \text{or} \quad y' = f(x, y)$$

(explicit form)

Usually has a general solution.

Direction
Field.



$$y' = y + x$$

General solution involves a constant C .

$$y' = 0 \Rightarrow y = C$$

In applications, we have to find a unique solution by determining a value of C from an initial condition. $y(x_0) = y_0$.

$$\begin{cases} F(x, y, y') = 0 \\ y(x_0) = y_0 \end{cases} \quad \text{IVP} \quad \text{initial value problem}$$

The solution is a particular solution.

(P3)

△ Separable ODE. (can't be used for higher order).

$$g(y) dy = f(x) dx.$$

$$\int g(y) dy = \int f(x) dx + C.$$

Ex: $y' = (x+1)e^{-x}y^2.$

soln: $y = \frac{1}{(x+2)e^{-x} - C}$

△ Exact ODE.

$$M(x,y) dx + N(x,y) dy = 0.$$

where $M dx + N dy$ is the differential

$$du = u_x dx + u_y dy.$$

Integrating factor: $F(x,y)$

$F(x,y) M(x,y) dx + F(x,y) N(x,y) dy = 0.$
is the differential $\underline{du} = 0.$

Ex: $y = xy'$ $\frac{-y dx + x dy}{x^2} = -\frac{y}{x^2} dx + \frac{1}{x} dy$
 $= d\left(\frac{y}{x}\right) = 0.$

(P4)

△ Linear ODE.

$$y' + p(x)y = r(x).$$

There ~~is~~ is general solution.

Homogeneous Linear ODE.

$$y' + p(x)y = 0.$$

$$\frac{dy}{y} = -p(x) dx.$$

$$\ln |y| = \int -p(x) dx + C.$$

$$y = c e^{-\int p(x) dx}.$$

△ Nonhomogeneous Linear ODE.

$$y' + p(x)y = r(x).$$

Find $F(x)$ such that:

$$\frac{d(Fy)}{dx} = Fy' + Fp y.$$

$$Fp = F' \Rightarrow F = e^{\int p dx} := e^h.$$

(P5)

$$(e^h y)' = r \cdot e^h.$$

$$e^h y = \int r e^h dx + C.$$

$$y = e^{-h} \left(\int r e^h dx + C \right)$$

$$= \underbrace{e^{-h} \int r e^h dx}_{\text{response to input } r} + \underbrace{C e^{-h}}_{\text{response to initial value.}}$$

(P6)

Second - order Linear ODE.

$$y'' + p(x)y' + q(x)y = r(x)$$

Linear in y .

$$\begin{cases} \text{homogeneous} & y'' + p(x)y' + q(x)y = 0 \\ \text{non homogeneous} & y'' + p(x)y' + q(x)y = \underbrace{r(x)}_{\neq 0} \end{cases}$$

Superposition principle

Any solu of nonhomogeneous

$$y = y_p + y_{gh}$$

↓
particular
solution
of nonhomo

↘
general
solution
of homogeneous.

△ homogeneous 2nd order ODE with constant coefficients

$$y'' + ay' + by = 0$$

Step 1 solve.

$$\lambda^2 + a\lambda + b = 0, \text{ for } \lambda$$

(P7)

Step 2.

Case	Type of Roots	General Soln
<u>I</u>	real $\lambda_1 \neq \lambda_2$.	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
<u>II</u>	Double $\lambda_1 = \lambda_2$.	$y = (C_1 + C_2 x) e^{\lambda x}$
<u>III</u>	Complex λ_1, λ_2 .	$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.

Type III: $\lambda = -\frac{1}{2}a \pm i\omega$.

$$y = C_1 e^{-\frac{1}{2}ax + i\omega x} + C_2 e^{-\frac{1}{2}ax - i\omega x}$$

$$= e^{-\frac{1}{2}ax} (C_1 e^{i\omega x} + C_2 e^{-i\omega x})$$

$$= e^{-\frac{1}{2}ax} (C_1 (\cos \omega x + i \sin \omega x) + C_2 (\cos \omega x - i \sin \omega x))$$

$$= e^{-\frac{1}{2}ax} (C_1 \cos \omega x + C_2 \sin \omega x)$$

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(P8).

△. Euler - Cauchy Equations.

$$x^2 y'' + ax y' + by = 0$$

Step 1.

Assume $y = x^m$

Step 2.

$$x^{\cancel{2}} m(m-1) x^m + a m x^m + b x^m = 0$$

$$m^2 + (a-1)m + b = 0$$

Step 3.

Case	Type of Roots	Solu.
I	$m_1 \neq m_2$ real	$y = c_1 x^{m_1} + c_2 x^{m_2}$
II	$m_1 = m_2$	$y = (c_1 + c_2 \ln x) x^m$
III	$m_1 \neq m_2$ complex	$y = c_1 x^{m_1} + c_2 x^{m_2}$

(P9)

△ Nonhomogeneous 2nd - order

Linear ODE, with constant coefficients.

$$y'' + ay' + by = r(x)$$

Term in $r(x)$	Choice for y_p
ke^{rx}	Ce^{rx}
kx^n	$K_n x^n + K_{n-1} x^{n-1} + \dots + K_1 x + K_0$
$k \cos wx$	} $K \cos wx + M \sin wx$
$k \sin wx$	
$ke^{ax} \cos wx$	} $e^{ax} (K \cos wx + M \sin wx)$
$ke^{ax} \sin wx$	

Ex: $y'' + 5y' + 4y = 10e^{-3x}$

(P10)

Higher order Linear ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \dots + p_1(x)y' + p_0(x)y = r(x)$$

Homogeneous with constant coefficient

$$y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1y' + a_0y = 0$$

Step 1 solve

$$\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$$

Step 2.

For distinct roots.

$$y = C_1 e^{\lambda_1 x} + \dots$$

For Double, triple, ... roots

$$y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x} + C_3 x^2 e^{\lambda_1 x} + \dots$$

Multiple roots

For double complex roots

$$y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_1 x}$$

(P.11)

Uniqueness of the Solution .

Solve $y'' + p(x)y' + q(x)y = 0$

Existence .

Complicated !