# ON THE CONTINUITY OF SCHUR-HORN MAPPING 

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#### Abstract

The Schur-Horn theorem is a well-known result that characterizes the relationship between the diagonal elements and eigenvalues of a symmetric (Hermitian) matrix. In this paper, we extend this theorem by exploring the eigenvalue perturbation of a symmetric (Hermitian) matrix with fixed diagonals, which is referred to as the continuity of the Schur-Horn mapping. We introduce a concept called strong Schur-Horn continuity, characterized by minimal constraints on the perturbation. We demonstrate that several categories of matrices exhibit strong Schur-Horn continuity. Leveraging this notion, along with a majorization constraint on the perturbation, we prove the Schur-Horn continuity for general symmetric (Hermitian) matrices. The Schur-Horn continuity finds applications in oblique manifold optimization related to quantum computing.


Key words. Schur-Horn, oblique manifold, mapping continuity, qOMM.

1. Introduction. Schur-Horn theorem was established in the mid-20th century. It continues to find applications and inspire advancements in various fields of mathematics and physics. Specifically, the characterization of eigenvalues and matrix diagonal entries continues to stimulate further research, driving advancements in quantum information theory, quantum optics, quantum metrology, spectral graph theory, convex optimization, and majorization theory. In this paper, we study the continuity of Schur-Horn mapping, which is adopted for our energy landscape analysis of objective functions in quantum computing [4].

Schur-Horn theorem is composed of two parts, as proved by Schur and Horn. We start by defining the condition known as the majorization.

Definition 1.1 (Majorization). Given vector $x \in \mathbb{R}^{n}$, notation $x^{\uparrow}$ denotes the reordered vector of $x$ with entries in non-decreasing order. Let $a$ and $\lambda$ be two vectors in $\mathbb{R}^{n}$. Vector $\lambda$ is majorized by a, denoted as $\lambda \prec a$, if

$$
\begin{equation*}
\sum_{i=1}^{k} \lambda_{i}^{\uparrow} \leq \sum_{i=1}^{k} a_{i}^{\uparrow}, \quad k=1, \cdots, n-1 ; \quad \text { and } \sum_{i=1}^{n} \lambda_{i}^{\uparrow}=\sum_{i=1}^{n} a_{i}^{\uparrow} . \tag{1.1}
\end{equation*}
$$

The majorization relation in (1.1) is equivalent to another commonly used definition in the literature, i.e., a vector $a$ is majorized by $\lambda$, if

$$
\sum_{i=1}^{k} a_{i}^{\downarrow} \leq \sum_{i=1}^{k} \lambda_{i}^{\downarrow}, \quad k=1, \cdots, n-1 ; \quad \text { and } \quad \sum_{i=1}^{n} a_{i}^{\downarrow}=\sum_{i=1}^{n} \lambda_{i}^{\downarrow}
$$

where notation $x^{\downarrow}$ denotes the reordered vector of $x$ with entries in non-increasing order. Then Schur-Horn theorem states:

- (Schur [19]) Let $H$ be a Hermitian matrix with eigenvalues $\lambda=\left(\lambda_{i}\right)_{1 \leq i \leq n}$ and diagonal entries $a=\left(a_{i i}\right)_{1 \leq i \leq n}$, then $\lambda \prec a$;
- (Horn [11]) If $a, \lambda \in \mathbb{R}^{n}$ satisfy $\lambda \prec a$, then there exists a symmetric (Hermitian) matrix $H$ whose diagonal entries are $a$ and eigenvalues are $\lambda$.
Schur-Horn theorem has been applied in various fields. In the realm of quantum optics and quantum state engineering, the Schur-Horn theorem has been applied

[^0]to design and manipulate desired quantum states [16, 20]. As for convex optimization, Schur-Horn theorem has implications in convex relaxations for graph and inverse eigenvalue problems [2]. The theorem provides constraints on the eigenvalues of positive semidefinite matrices, enabling the formulation of optimization problems with eigenvalue constraints and facilitating the development of efficient algorithms for solving such problems.

There has been a rich history in proving the Schur-Horn theorem, specifically the Horn part. In general, proofs could be grouped into nonconstructive ones [5, 11, 12] and constructive ones [3, 23]. Chu [5] utilized an optimization-based limiting process to prove the existence of the matrix in the Horn part. Leite et al. [12] gave an algebraic proof, which could be extended to analogous results for skew-symmetric matrices as well. Constructive proofs [3, 23] were based on Givens rotation and could be viewed as an algorithm for constructing $H$ in Horn part given $a$ and $\lambda$ satisfying the majorization condition. Generalizations of constructive algorithms can be found in [6, 7]. Recently, Matthew Fickus et al. [8] proposed an algorithm based on the finite frame theory to procedure every example of the matrix in the Horn part.
1.1. Contribution. Schur-Horn theorem establishes the connections among diagonal entries, eigenvalues, and a symmetric matrix. In the following, we first define a Schur-Horn mapping based on the Schur-Horn theorem and then prove the continuity of the mapping.

Given a target diagonal vector $d \in \mathbb{R}^{n}$, we define two sets of matrices ${ }^{1}$

$$
S_{d}=\left\{\Lambda \in \mathbb{R}^{n \times n}: \Lambda \text { diagonal }, \operatorname{diag}(\Lambda) \prec d\right\}
$$

and

$$
M_{d}=\left\{A \in \mathbb{R}^{n \times n}: \operatorname{diag}(A)=d, A=A^{\top}\right\}
$$

One can define an equivalence relation between two matrices $A_{1}$ and $A_{2}$ over $M_{d}$ as

$$
\begin{equation*}
A_{1} \sim A_{2} \text { if } A_{1}, A_{2} \text { have the same eigenvalues. } \tag{1.2}
\end{equation*}
$$

Then, we can define a mapping between $S_{d}$ and the quotient space of $M_{d}$ with the equivalence relation (1.2), whose existence is guaranteed by the Schur-Horn theorem,

$$
\begin{align*}
F: & S_{d} \rightarrow M_{d} / \sim \\
& \Lambda \mapsto\left[Q \Lambda Q^{\top}\right], \quad Q \text { is an orthogonal matrix such that } \operatorname{diag}\left(Q \Lambda Q^{\top}\right)=d \tag{1.3}
\end{align*}
$$

The mapping $F$ is called the Schur-Horn mapping. Furthermore, we introduce the Hausdorff distance with the Frobenius norm $\|\cdot\|_{F}$, i.e.,

$$
\begin{equation*}
d_{\mathrm{H}}\left(\left[A_{1}\right],\left[A_{2}\right]\right):=\max \left\{\sup _{X \in\left[A_{1}\right]} \inf _{Y \in\left[A_{2}\right]}\|X-Y\|_{\mathrm{F}}, \sup _{Y \in\left[A_{2}\right]} \inf _{X \in\left[A_{1}\right]}\|X-Y\|_{\mathrm{F}}\right\} \tag{1.4}
\end{equation*}
$$

This Hausdorff distance measures the distance between two elements in $M_{d} / \sim$. We also remark that actually the sup and inf in (1.4) can be replaced by max and min respectively since $\left[A_{1}\right]$ and $\left[A_{2}\right]$ are both compact sets in $M_{d}$ and $f(Y)=\|X-Y\|_{\mathrm{F}}$

[^1]and $g(X)=\min _{Y \in\left[A_{2}\right]}\|X-Y\|_{\mathrm{F}}$ are continuous functions. Indeed, Hausdorff distance is a metric over the set of compact subsets [21]. With the Hausdorff distance being a properly defined metric in $M_{d} / \sim$, we can claim the continuity of the SchurHorn mapping $F(\cdot)$, i.e., if $\Lambda_{1}, \Lambda_{2} \in S_{d}$ are sufficiently close, then $F\left(\Lambda_{1}\right), F\left(\Lambda_{2}\right)$ can be close enough under the Hausdorff distance. Rigorously, we will first establish a perturbative analysis for $F(\cdot)$ in Theorem 1.3 and then state the continuity of the Schur-Horn mapping in Corollary 1.4, which are the main contributions of this paper.

For ease of expression and reference later, we introduce the definition of SchurHorn continuity as follows.

Definition 1.2 (Schur-Horn Continuity). Suppose $A$ is an $n$-by-n real symmetric (complex Hermitian) matrix with eigenvalues $\lambda \in \mathbb{R}^{n}$ and diagonal entries $d \in \mathbb{R}^{n}$. Let $\tilde{\lambda}$ be a perturbation of $\lambda$ such that $\|\lambda-\tilde{\lambda}\|_{2}=O(\varepsilon)$ for $\varepsilon>0$ sufficiently small and $\tilde{\lambda} \prec d$. Then, there exists a real symmetric (complex Hermitian) matrix $\widetilde{B}$ with eigenvalues $\tilde{\lambda}$ and diagonal entries d such that $\|A-\widetilde{B}\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$.

Theorem 1.3. Any symmetric matrix $A \in \mathbb{R}^{n \times n}$ is Schur-Horn continuous.
From Theorem 1.3, one could easily deduce the continuity of the Schur-Horn mapping as the following corollary.

Corollary 1.4. Schur-Horn mapping $F$ is a continuous mapping from $S_{d}$ to $M_{d} / \sim$ with Hausdorff distance $d_{H}$.

Proof. Given $\Lambda_{1}, \Lambda_{2} \in S_{d}$ such that $\left\|\Lambda_{1}-\Lambda_{2}\right\|_{F}=O(\varepsilon)$, denote $\left[A_{1}\right]=F\left(\Lambda_{1}\right)$, $\left[A_{2}\right]=F\left(\Lambda_{2}\right)$. According to Theorem 1.3 we have $\min _{Y \in\left[A_{2}\right]}\|X-Y\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$. Note that $g(X)=\min _{Y \in\left[A_{2}\right]}\|X-Y\|_{\mathrm{F}}$ is a continuous function and $\left[A_{1}\right]$ is a compact set, it yields that

$$
\max _{X \in\left[A_{1}\right]} \min _{Y \in\left[A_{2}\right]}\|X-Y\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right) .
$$

Similarly, we have

$$
\max _{Y \in\left[A_{2}\right]} \min _{X \in\left[A_{1}\right]}\|X-Y\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right) .
$$

Thus, from the definition of Hausdorff distance (1.4) we obtain the continuity of Schur-Horn mapping.

For some applications where matrices are Hermitian, we can still define the SchurHorn mapping and prove its continuity. Consider

$$
N_{d}=\left\{A \in \mathbb{C}^{n \times n}: \operatorname{diag}(A)=d, A=A^{*}\right\}
$$

we can define Schur-Horn mapping for Hermitian scenario as

$$
\begin{align*}
F: & S_{d} \rightarrow N_{d} / \sim \\
& \Lambda \mapsto\left[Q \Lambda Q^{*}\right], \quad Q \text { is a unitary matrix such that } \operatorname{diag}\left(Q \Lambda Q^{*}\right)=d \tag{1.5}
\end{align*}
$$

Here, we abuse the notation $F$ to denote the Schur-Horn mapping for Hermitian matrices. Then we have similar results as follows.

Theorem 1.5. Any Hermitian matrix $A \in \mathbb{C}^{n \times n}$ is Schur-Horn continuous.
Corollary 1.6. Schur-Horn mapping $F$ is a continuous mapping from $S_{d}$ to $N_{d} / \sim$ with Hausdorff distance $d_{H}$.
1.2. Applications. The continuity property of the Schur-Horn mapping is useful in analyzing the manifold optimization problems. For example, consider the landscape analysis of an objective function over the oblique manifold. The descent direction of such a problem has to incorporate the manifold information, and the perturbative analysis of a stationary point on the manifold directly links to the continuity of the Schur-Horn mapping. We provide a concrete application of the continuity of the Schur-Horn mapping.

Given a negative definite Hermitian matrix $A \in \mathbb{C}^{n \times n}$, we consider the following manifold optimization problem,

$$
\begin{equation*}
\min _{X \in \mathcal{O B}(n, p)} E_{0}(X)=\operatorname{tr}\left(\left(2 I-X^{*} X\right) X^{*} A X\right) \tag{1.6}
\end{equation*}
$$

where the oblique manifold is defined as,

$$
\begin{equation*}
\mathcal{O B}(n, p)=\left\{X \in \mathbb{C}^{n \times p} \mid \operatorname{diag}\left(X^{*} X\right)=\mathbf{1}\right\} \tag{1.7}
\end{equation*}
$$

where 1 denotes an all-one vector of length $p$. The minimization problem (1.6) without the oblique manifold constraint has been known as the unconstrained orbital minimization method (OMM) [17, 18] in the literature, which is used to seek the low-lying eigenpairs of $A$. In [1], the OMM objective function is adopted in a variational quantum eigensolver (VQE) on quantum computers, known as quantum orbital minimization method (qOMM). The manifold optimization problem (1.6) is the optimization problem of $q \mathrm{OMM}$, where the oblique manifold constraint appears due to the unitary quantum state constraint from the quantum computer.

Without the oblique manifold constraint, OMM has an attractive property: all minima are formed by the eigenvectors of $A$ corresponding to the low-lying eigenvalues, and it has no spurious local minima [15]. With the oblique manifold constraint, we would like to have the same property. In the study of the first-order stationary points of (1.6), we would like to show that some of them are strict saddle points, and a local perturbation leads to decay in the objective function. However, the oblique manifold constraint requires that the perturbed points have to stay in the manifold, i.e., $\operatorname{diag}\left(X^{*} X\right)=1$. The continuity of the Schur-Horn mapping assures that for any local perturbation on eigenvalues of $X^{*} X$ there is a corresponding point in the neighborhood of $X^{*} X$ on the manifold. Then, the objective function at a saddle point decays for a particular perturbation, and hence, the same property of OMM holds for qOMM.

Recently, other objective functions [9, 10, 13, 14, 22] have been applied in VQE on quantum computers and lead to the following constraint optimization problems,

$$
\min _{X \in \mathcal{O B}(n, p)} \frac{1}{2} \operatorname{tr}\left(X^{*} A X\right)+\frac{\mu}{4}\left\|X^{*} X-I\right\|_{\mathrm{F}}^{2} \quad \text { and } \min _{X \in \mathcal{O B}(n, p)} \frac{1}{2}\left\|X X^{*}-A\right\|_{\mathrm{F}}^{2}
$$

In their landscape analysis, the continuity of the Schur-Horn mapping can be applied to similar scenarios of saddle point analysis. Similar property as that of OMM could be proved.
1.3. Organization. The rest of the paper is organized as follows. In section 2, we first establish the Schur-Horn continuity for real diagonal matrices. Then, in section 3, we introduce the concept of strong Schur-Horn continuity and demonstrate its application to specific matrix types, which plays a crucial role in proving our main theorem. In section 4, we combine the previous results to prove the Schur-Horn continuity for symmetric matrices. The Schur-Horn continuity for Hermitian matrices is covered in section 5 . Finally, we conclude the paper in section 6 .
2. Schur-Horn Continuity of Diagonal Matrices. We first prove the SchurHorn continuity of diagonal matrices, as in Theorem 2.2. The proof explicitly shows that the majorization condition plays an essential role in the Schur-Horn continuity.

Before proving Theorem 2.2, we first prove Lemma 2.1, which is the generalized version of Schur-Horn continuity for matrices of size $2 \times 2$. Lemma 2.1 will be repeatedly used throughout this paper. In the following, standard big $O$ and $\operatorname{big} \Theta$ notations are used with $\varepsilon$ being the asymptotic variable approaching zero. Other variables, including matrix dimensions and matrix nonzero entries, are viewed as constants.

Lemma 2.1. Given $\varepsilon>0$ sufficiently small and $d_{1}, d_{2} \in \mathbb{R}$. Let $B$ be a symmetric matrix of form,

$$
B=\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{array}\right]=\left[\begin{array}{cc}
d_{1}-f(\varepsilon) & b_{12} \\
b_{12} & d_{2}+g(\varepsilon)
\end{array}\right]
$$

with $f(\varepsilon)=\Theta\left(\varepsilon^{\alpha}\right), g(\varepsilon)=\Theta\left(\varepsilon^{\beta}\right)$ for $\alpha, \beta>0$. Further, we assume that

$$
b_{12}^{2}+f(\varepsilon)\left(d_{2}-d_{1}+g(\varepsilon)\right) \geq 0
$$

Then there exists a Givens rotation $G$ with rotation angle $\theta=\Theta\left(\varepsilon^{\gamma}\right)$ such that the $(1,1)$ entry of $\widetilde{B}=G B G^{\top}$ is $d_{1}$ and $\|\widetilde{B}-B\|_{\mathrm{F}}=O\left(\varepsilon^{\delta}\right)$, where various scenarios of $\gamma$ and $\delta$ are provided in Table 1.

| Various Scenarios |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{12} \neq 0$ | $\gamma$ | $\delta$ |  |  |  |
| $b_{12}=0$ | $d_{1} \neq d_{2}$ | $\alpha$ | $\alpha$ |  |  |
| $b_{12}=0$ | $d_{1}=d_{2}$ | $\alpha>\beta$ | $(\alpha-\beta) / 2$ |  |  |
| $b_{12}=0$ | $d_{1}=d_{2}$ | $\alpha \leq \beta$ | 0 |  |  |
| TABLE 1 |  |  |  |  | $\alpha / 2$ |
| T |  |  |  |  | $\alpha$ |

Various scenarios of $b_{12}, d_{1}, d_{2}, \gamma$, and $\delta$ for Lemma 2.1.

Proof. Denote the Givens rotation matrix as $G=\left[\begin{array}{cc}c & s \\ -s & c\end{array}\right]$, where $c=\cos \theta$ and $s=\sin \theta$. Then we have

$$
\widetilde{B}=G B G^{\top}=\left[\begin{array}{cc}
c^{2} b_{11}+s^{2} b_{22}+2 c s b_{12} & \left(c^{2}-s^{2}\right) b_{12}+c s\left(b_{22}-b_{11}\right)  \tag{2.1}\\
\left(c^{2}-s^{2}\right) b_{12}+c s\left(b_{22}-b_{11}\right) & c^{2} b_{22}+s^{2} b_{11}-2 c s b_{12}
\end{array}\right]
$$

Equating the $(1,1)$ entry of $\widetilde{B}$ and $d_{1}$, we obtain,

$$
\begin{equation*}
c^{2} b_{11}+s^{2} b_{22}+2 c s b_{12}=d_{1} \tag{2.2}
\end{equation*}
$$

Denoting $t=\frac{s}{c}=\tan \theta$, we reformulate (2.2) into

$$
\begin{equation*}
\left(b_{22}-d_{1}\right) t^{2}+2 b_{12} t+b_{11}-d_{1}=0 \tag{2.3}
\end{equation*}
$$

Equation (2.3) has real solutions if and only if

$$
\begin{equation*}
\Delta=4\left(b_{12}^{2}-\left(b_{22}-d_{1}\right)\left(b_{11}-d_{1}\right)\right)=4\left(b_{12}^{2}+f(\varepsilon)\left(d_{2}-d_{1}+g(\varepsilon)\right)\right) \geq 0 \tag{2.4}
\end{equation*}
$$

According to the assumption in the lemma, we know that (2.3) always has at least one solution, thus $G$ exists. Furthermore, by solving the quadratic equation, one has

$$
\begin{equation*}
t=\frac{f(\varepsilon)}{b_{12} \pm \sqrt{b_{12}^{2}+f(\varepsilon)\left(d_{2}-d_{1}+g(\varepsilon)\right)}} \tag{2.5}
\end{equation*}
$$

Now, we divide the discussion into various scenarios, as in Table 1.
Case 1: $b_{12} \neq 0$.
In this case, $\Delta$ is positive when $\varepsilon$ is sufficiently small. The quadratic equation (2.3) then has two solutions in (2.5), and one of them is of order $\frac{f(\varepsilon)}{2 b_{12}}$. Hence, one solution gives $\tan \theta=\Theta\left(\varepsilon^{\alpha}\right)$, thus $\theta=\Theta\left(\varepsilon^{\alpha}\right)$. By the triangular inequality of matrix norm, we obtain,

$$
\begin{aligned}
\|\widetilde{B}-B\|_{\mathrm{F}} & \leq\left\|G B G^{\top}-G B\right\|_{\mathrm{F}}+\|G B-B\|_{\mathrm{F}} \\
& \leq\|G B\|_{\mathrm{F}} \cdot\left\|G^{\top}-I\right\|_{\mathrm{F}}+\|G-I\|_{\mathrm{F}} \cdot\|B\|_{\mathrm{F}}=O\left(\varepsilon^{\alpha}\right)
\end{aligned}
$$

Case 2: $b_{12}=0$ and $d_{2} \neq d_{1}$.
When $b_{12}=0$, we have

$$
\widetilde{B}-B=\left[\begin{array}{cc}
f(\varepsilon) & c s\left(d_{2}-d_{1}+g(\varepsilon)+f(\varepsilon)\right) \\
c s\left(d_{2}-d_{1}+g(\varepsilon)+f(\varepsilon)\right) & -f(\varepsilon)
\end{array}\right]
$$

At this time, the solutions (2.5) of the quadratic equation (2.3) admit,

$$
t= \pm \sqrt{\frac{f(\varepsilon)}{d_{2}-d_{1}+g(\varepsilon)}}=\Theta\left(\varepsilon^{\alpha / 2}\right)
$$

where the positivity of the quantity under the square root is guaranteed by (2.4). Thus, we can derive $\theta=\Theta\left(\varepsilon^{\alpha / 2}\right)$ and $\|\widetilde{B}-B\|_{\mathrm{F}}=O\left(\varepsilon^{\alpha / 2}\right)$.

Case 3 and Case 4: $b_{12}=0$ and $d_{2}=d_{1}$.
Denoting $d=d_{1}=d_{2}$, the solutions (2.5) of the quadratic equation (2.3) admit,

$$
t= \pm \sqrt{\frac{f(\varepsilon)}{g(\varepsilon)}}=\Theta\left(\varepsilon^{(\alpha-\beta) / 2}\right)
$$

where the positivity of the quantity under the square root is also guaranteed by (2.4). Additionally, we have

$$
\widetilde{B}-B=\left[\begin{array}{cc}
f(\varepsilon) & c s(g(\varepsilon)+f(\varepsilon))  \tag{2.6}\\
c s(g(\varepsilon)+f(\varepsilon)) & -f(\varepsilon)
\end{array}\right]
$$

where $|s| \leq 1$ and $|c| \leq 1$.
When $\alpha>\beta$ and $\varepsilon$ sufficiently small, we have $\theta=\Theta\left(\varepsilon^{(\alpha-\beta) / 2}\right)$. However, consider entries comparison of $\widetilde{B}$ and $B$ as (2.6), one can deduce that

$$
c s(g(\varepsilon)+f(\varepsilon))=\Theta\left(\varepsilon^{(\alpha-\beta) / 2+\beta}\right)=\Theta\left(\varepsilon^{(\alpha+\beta) / 2}\right)
$$

with $f(\varepsilon)=\Theta\left(\varepsilon^{\alpha}\right)$. Note that $\alpha>\beta$ implies $\frac{\alpha+\beta}{2}<\alpha$, thus we have $\|\widetilde{B}-B\|_{\mathrm{F}}=$ $O\left(\varepsilon^{(\alpha+\beta) / 2}\right)$.

When $\alpha \leq \beta$, we have $\theta=\Theta(1)$. Comparing entries of $\widetilde{B}$ and $B$ as (2.6), we could see that the norm of $\widetilde{B}-B$ is bounded by the lower order of $f(\varepsilon)$ and $g(\varepsilon)$, and we have $\|\widetilde{B}-B\|_{\mathrm{F}}=O\left(\varepsilon^{\alpha}\right)$.

Theorem 2.2. Any diagonal matrix $A \in \mathbb{R}^{n \times n}$ is Schur-Horn continuous.
Proof. Without loss of generality, we assume that the diagonal entries of $A$ are non-decreasing, i.e., $d_{1} \leq d_{2} \leq \cdots \leq d_{n}$. The perturbed matrix is denoted as $\widetilde{A}^{(0)}$. The perturbed eigenvalues are denoted as $\tilde{\lambda}_{i}=d_{i}+h_{i}(\varepsilon)$. When the eigenvalues have a gap, i.e., $\lambda_{i}<\lambda_{i+1}$, the perturbed eigenvalues keep the ordering, i.e., $\tilde{\lambda}_{i} \leq \tilde{\lambda}_{i+1}$ for sufficiently small $\varepsilon$. When eigenvalues are identical, $\lambda_{i}=\lambda_{i+1}$, the perturbed eigenvalues are ordered based on their perturbations. Then we have the majorization relations given $\varepsilon>0$ sufficiently small

$$
\begin{gathered}
\tilde{\lambda}_{1} \leq d_{1} \\
\tilde{\lambda}_{1}+\tilde{\lambda}_{2} \leq d_{1}+d_{2} \\
\vdots \\
\tilde{\lambda}_{1}+\cdots+\tilde{\lambda}_{n-1} \leq d_{1}+\cdots+d_{n-1} \\
\tilde{\lambda}_{1}+\cdots+\tilde{\lambda}_{n-1}+\tilde{\lambda}_{n}=d_{1}+\cdots+d_{n-1}+d_{n}
\end{gathered}
$$

Hence we have

$$
\begin{aligned}
& h_{1}(\varepsilon)+h_{2}(\varepsilon)+\cdots+h_{i}(\varepsilon) \leq 0, \quad i=1, \cdots, n-1, \text { and } \\
& h_{1}(\varepsilon)+\cdots+h_{n}(\varepsilon)=0 .
\end{aligned}
$$

Next, we describe a procedure to correct the diagonal entries from perturbed $\tilde{\lambda}_{i}$ to $d_{i}$.
We maintain a priority queue with diagonal indices as elements. For any diagonal index $i$ in the queue, we ensure that $d_{i}+\tilde{h}_{i}(\varepsilon)$ has a negative perturbation $\tilde{h}_{i}(\varepsilon)<0$, where $\tilde{h}_{i}(\varepsilon)$ denotes the updated perturbation throughout the procedure. Starting from the first diagonal entry, we check and enqueue the index $i=1,2, \ldots$ in order if $\tilde{h}_{i}(\varepsilon)<0$ and skip the index $i$ if $\tilde{h}_{i}(\varepsilon)_{\tilde{\sim}}=0$. We keep on checking and enqueuing indices until the first index $j$ such that $\tilde{h}_{j}(\varepsilon)>0$. If $j$ does not exist, then by the last equation in majorization relation, we know that the queue is also empty and the diagonal entries of the perturbed $A$ have all been corrected. Otherwise, we obtain a $j$ and the updated perturbations satisfy,

$$
\begin{equation*}
\tilde{h}_{1}(\varepsilon)+\cdots+\tilde{h}_{j}(\varepsilon)=h_{1}(\varepsilon)+\cdots+h_{j}(\varepsilon) \leq 0 . \tag{2.7}
\end{equation*}
$$

This condition is satisfied in the first correction step and we will verify it after each step. By the $j$-th majorization relation, the queue is guaranteed to be non-empty. We pop an index from the queue and denote it as $i$. The current working matrix is denoted as $\widetilde{A}^{(i-1)} .^{2}$

Based on the property of $i$ and $j$, the inequality assumption in Lemma 2.1 is always satisfied. Hence, the lemma provides a Givens rotation applying to the $i$-th and $j$ th columns and rows to correct the $(i, i)$ diagonal entry. Without loss of generality, applying the extended Givens rotation matrix symmetrically to the current matrix

[^2]$\widetilde{A}^{(i-1)}$ we obtain $\widetilde{A}^{(i)}$, whose top-left $j$-by- $j$ submatrix admits,
\[

$$
\begin{align*}
& {\left[\begin{array}{ccccc}
I & & & \\
& c & & s \\
& & I & \\
& -s & & c
\end{array}\right]\left[\begin{array}{cccc}
\widetilde{A}_{\mathcal{I}_{1} \mathcal{I}_{1}} & p_{i \mathcal{I}_{1}}^{\top} & \widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{1}}^{\top} & p_{j \mathcal{I}_{1}}^{\top} \\
p_{i \mathcal{I}_{1}} & d_{i}+\tilde{h}_{i}(\varepsilon) & p_{\mathcal{I}_{2} i}^{\top} & 0 \\
\widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{1}} & p_{\mathcal{I}_{2} i} & \widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{2}} & p_{j \mathcal{I}_{2}}^{\top} \\
p_{j \mathcal{I}_{1}} & 0 & p_{j \mathcal{I}_{2}} & d_{j}+\tilde{h}_{j}(\varepsilon)
\end{array}\right]\left[\begin{array}{cccc}
I & & \\
& c & & -s \\
& & I & \\
& s & c
\end{array}\right]} \\
& =\left[\begin{array}{cccc}
\widetilde{A}_{\mathcal{I}_{1} \mathcal{I}_{1}} & c p_{i \mathcal{I}_{1}}^{\top}+s p_{j \mathcal{I}_{1}}^{\top} & \widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{1}}^{\top} & c p_{j \mathcal{I}_{1}}^{\top}-s p_{i \mathcal{I}_{1}}^{\top} \\
c p_{i \mathcal{I}_{1}}+s p_{j \mathcal{I}_{1}} & d_{i} & c p_{\mathcal{I}_{2} i}^{\top}+s p_{j \mathcal{I}_{2}} & p_{i j} \\
\widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{1}} & c p_{\mathcal{I}_{2} i}+s p_{j \mathcal{I}_{2}}^{\top} & \widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{2}} & c p_{j \mathcal{I}_{2}}^{\top}-s p_{\mathcal{I}_{2} i} \\
c p_{j \mathcal{I}_{1}}-s p_{i \mathcal{I}_{1}} & p_{j i} & c p_{j \mathcal{I}_{2}}-s p_{\mathcal{I}_{2} i}^{\top} & d_{j}+\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)
\end{array}\right], \tag{2.8}
\end{align*}
$$
\]

where $\mathcal{I}_{1}=\{1, \ldots, i-1\}, \mathcal{I}_{2}=\{i+1, \ldots, j-1\}$, vectors $p_{i \mathcal{I}_{1}}, p_{\mathcal{I}_{2} i}, p_{j \mathcal{I}_{1}}$ and $p_{j \mathcal{I}_{2}}$ are all perturbations introduced in previous steps, submatrices $\widetilde{A}_{\mathcal{I}_{1} \mathcal{I}_{1}}, \widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{1}}$, and $\widetilde{A}_{\mathcal{I}_{2} \mathcal{I}_{2}}$ are untouched submatrices of $\widetilde{A}^{(i-1)},{ }^{3}$ and

$$
p_{i j}=p_{j i}=c s\left(d_{j}-d_{i}+\tilde{h}_{j}(\varepsilon)-\tilde{h}_{i}(\varepsilon)\right)
$$

The Givens rotation above corrects the $(i, i)$ diagonal entry and it only changes the top-left $j$-by- $j$ submatrix of $\widetilde{A}^{(i-1)}$, leaving the remain part of $\widetilde{A}^{(i-1)}$ unchanged. Since $\|\lambda-\tilde{\lambda}\|_{2}=O(\varepsilon)$, from the correction procedure we have

$$
\begin{equation*}
\tilde{h}_{i}(\varepsilon)=O(\varepsilon)=\Theta\left(\varepsilon^{\alpha}\right), \quad \tilde{h}_{j}(\varepsilon)=O(\varepsilon)=\Theta\left(\varepsilon^{\beta}\right) \tag{2.9}
\end{equation*}
$$

with $\alpha \geq 1$ and $\beta \geq 1$.
We now have three cases: i) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)<0$; ii) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)=0$; and iii) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)>0$. In case i), we enqueue $j$ and start checking the following indices. In case ii), we skip $j$ and start checking the indices after $j$. In case iii), we pop another index from the queue and repeat the correction procedure. In all cases, the updated perturbation at index $i$ and $j$ are 0 and $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)$, respectively. Hence (2.7) holds for all indices greater or equal to $j$. Then, the majorization relations guarantee that the procedure ends if and only if all diagonal entries have been corrected.

Finally, we show that the corrected matrix is within an $\varepsilon^{1 / 2}$ neighborhood of the original matrix. In the above procedure, each step corrects at least one diagonal index, and the procedure finishes in at most $n$ steps for $n$ being the matrix size. Now, we show that all the off-diagonals of $\widetilde{A}^{(i-1)}$ are $O\left(\varepsilon^{1 / 2}\right)$ for $i=1, \ldots, n$, by induction. It is obvious that all the off-diagonals of $\widetilde{A}^{(0)}=\operatorname{diag}(\tilde{\lambda})$ are zeros and hence $O\left(\varepsilon^{1 / 2}\right)$, which gives the start point of induction. For those $\tilde{h}_{i}(\varepsilon)=0, \widetilde{A}^{(i)}=\widetilde{A}^{(i-1)}$, hence we only need to consider the case when $\tilde{h}_{i}(\varepsilon) \neq 0$. Note that vectors $p_{i \mathcal{I}_{1}}, p_{\mathcal{I}_{2} i}, p_{j \mathcal{I}_{1}}$ and $p_{j \mathcal{I}_{2}}$ in (2.8) are $O\left(\varepsilon^{1 / 2}\right)$ by our induction assumption, it implies that the off-diagonals of $\widetilde{A}^{(i)}$ are $O\left(\varepsilon^{1 / 2}\right)$ except $p_{i j}$ and $p_{j i}$. Denote

$$
B=\left[\begin{array}{cc}
d_{i}+\tilde{h}_{i}(\varepsilon) & 0 \\
0 & d_{j}+\tilde{h}_{j}(\varepsilon)
\end{array}\right], \quad \widetilde{B}=\left[\begin{array}{cc}
d_{i} & p_{i j} \\
p_{j i} & d_{j}+\tilde{h}_{j}(\varepsilon)+\tilde{h}_{i}(\varepsilon)
\end{array}\right]
$$

we split the discussion according to the scenarios with $b_{12}=0$ in Lemma 2.1 and (2.9) as follow:

[^3](i) $d_{i} \neq d_{j},\|B-\widetilde{B}\|_{\mathrm{F}}=O\left(\varepsilon^{\alpha / 2}\right)=O\left(\varepsilon^{1 / 2}\right)$;
(ii) $d_{i}=d_{j}$ and $\alpha>\beta,\|B-\widetilde{B}\|_{\mathrm{F}}=O\left(\varepsilon^{(\alpha+\beta) / 2}\right)=O\left(\varepsilon^{1 / 2}\right)$;
(iii) $d_{i}=d_{j}$ and $\alpha \leq \beta,\|B-\widetilde{B}\|_{\mathrm{F}}^{\mathrm{F}}=O\left(\varepsilon^{\alpha}\right)=O\left(\varepsilon^{1 / 2}\right)$.

Therefore, we conclude that all the off-diagonals of $\widetilde{A}^{(i)}$ are $O\left(\varepsilon^{1 / 2}\right)$. Furthermore, note that the diagonals of $\widetilde{A}^{(i)}$ and $\widetilde{A}^{(i-1)}$ only differ at indices $i$ and $j$, together with $\|B-\widetilde{B}\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$, it implies that $\left\|\widetilde{A}^{(i)}-\widetilde{A}^{(i-1)}\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$. By triangular inequality of Frobenius norm, the distance between the corrected matrix and the original matrix is bounded as,

$$
\left\|A-\widetilde{A}^{(n)}\right\|_{\mathrm{F}} \leq\left\|A-\widetilde{A}^{(0)}\right\|_{\mathrm{F}}+\sum_{k=1}^{n}\left\|\widetilde{A}^{(k-1)}-\widetilde{A}^{(k)}\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right) .
$$

Thus, $A$ is Schur-Horn continuous.
Remark 2.3. Theorem 2.2 is covered by our main Theorem 1.5. And the proof of Theorem 2.2 has the same structure as that of Theorem 1.5, where more complicated scenarios are discussed in the later one. We present the diagonal matrix case as a stand-alone theorem to facilitate the understanding of the main theorem proof. The diagonal matrix perturbation is also used to prove the strong Schur-Horn continuity.
3. Strong Schur-Horn Continuity. In this section, we define the strong SchurHorn continuity. The strong Schur-Horn continuity is a stronger version of the SchurHorn continuity, which plays a key role in the proof of Schur-Horn continuity of general symmetric matrices. And we will prove that if a matrix is strongly SchurHorn continuous then it is Schur-Horn continuous, but not the other way around. Before delving into the proofs, it is essential to define the spectrum window and strong Schur-Horn continuity.

Definition 3.1 (Spectrum Window). Let $A$ be a symmetric matrix. The spectrum window of $A$ is defined as the closed interval of the minimum and maximum eigenvalues of $A$, i.e., ${ }^{4}$

$$
\omega(A):=\left[\lambda_{\min }(A), \lambda_{\max }(A)\right] .
$$

Definition 3.2 (Strong Schur-Horn Continuity). Suppose $A \in \mathbb{R}^{n \times n}$ is a symmetric matrix with an eigendecomposition $A=Q \Lambda Q^{\top}$, where $Q$ is the orthonormal eigenvector matrix and $\Lambda$ is the diagonal eigenvalue matrix. Matrix $A$ is strongly Schur-Horn continuous if, for any perturbed eigenvalues $\widetilde{\Lambda}$ satisfying $\operatorname{tr}(\widetilde{\Lambda})=\operatorname{tr}(\Lambda)$ and $\|\widetilde{\Lambda}-\Lambda\|_{\mathcal{F}}=O(\varepsilon)$ for $\varepsilon>0$ sufficiently small, there exists a symmetric matrix $\widetilde{B}=G_{2} Q G_{1} \widetilde{\Lambda} G_{1}^{\top} Q^{\top} G_{2}^{\top}$ such that

1. $\operatorname{diag}(\widetilde{B})=\operatorname{diag}(A)$,
2. $G_{1}$ and $G_{2}$ are orthogonal matrices, and
3. $\left\|G_{i}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i=1,2$.

With relaxed requirements, it is straightforward to conclude that the strong SchurHorn continuity directly implies the Schur-Horn continuity.

[^4]Corollary 3.3. If a matrix is strongly Schur-Horn continuous, then it is SchurHorn continuous.

Remark 3.4. There exists a counterexample matrix $A$ being Schur-Horn continuous but not strongly Schur-Horn continuous. Consider the matrix $A$ and its perturbed eigenvalue matrix $\widetilde{\Lambda}$,

$$
A=\left[\begin{array}{ll}
1 & 0 \\
0 & 2
\end{array}\right] \text { and } \widetilde{\Lambda}=\left[\begin{array}{cc}
1+\varepsilon & 0 \\
0 & 2-\varepsilon
\end{array}\right]
$$

where $\varepsilon>0$ is sufficiently small, and the perturbation satisfies the last majorization relation. By Theorem 2.2, the diagonal matrix $A$ is Schur-Horn continuous. However, for the given perturbation above, the first majorization relation is violated. By Schur theorem, there does not exist a matrix $\widetilde{B}$ whose eigenvalue matrix is $\widetilde{\Lambda}$ and diagonal entries being 1 and 2. Hence, we conclude that $A$ is not strongly Schur-Horn continuous.

Comparing Definition 3.2 and Definition 1.2, there are two differences. First, the requirement of the perturbation is relaxed in Definition 3.2. In Definition 1.2, the perturbation needs to satisfy all majorization relations as in (1.1). While, Definition 3.2 only requires the perturbation to satisfy the last majorization relation. This difference is essential between the Schur-Horn continuity and the strong Schur-Horn continuity. Furthermore, we have Proposition 3.5 describing the relation of eigenvalues and diagonal entries of a strongly Schur-Horn continuous matrix, which directly extends the result of the Schur part in the Schur-Horn theorem. The proof of Proposition 3.5 can be found in Appendix A.

Proposition 3.5. Suppose matrix $A \in \mathbb{R}^{n \times n}$ is strongly Schur-Horn continuous with eigenvalues $\lambda \in \mathbb{R}^{n}$ and diagonal entries $d \in \mathbb{R}^{n}$. Without loss of generality, both $\lambda$ and $d$ are in non-decreasing order, then either $A$ is a scalar matrix, i.e.,

$$
\lambda_{1}=\cdots=\lambda_{n}=d_{1}=\cdots=d_{n}
$$

or the first $n-1$ majorization inequalities of $A$ are strict, i.e.,

$$
\begin{gather*}
\lambda_{1}<d_{1} \\
\lambda_{1}+\lambda_{2}<d_{1}+d_{2} \\
\vdots  \tag{3.1}\\
\lambda_{1}+\cdots+\lambda_{n-1}<d_{1}+\cdots+d_{n-1} \\
\lambda_{1}+\cdots+\lambda_{n-1}+\lambda_{n}=d_{1}+\cdots+d_{n-1}+d_{n}
\end{gather*}
$$

The second difference is the transformation of $\widetilde{\Lambda}$, i.e., the eigenvector matrix of $\widetilde{B}$. In Definition 1.2 , no restriction is applied to the eigenvector matrix of $\widetilde{B}$, whereas in Definition 3.2, the eigenvector matrix is required to be an $\varepsilon$ perturbation of the original eigenvector matrix of $A$. The second difference is not essential. The explicit expression $G_{2} Q G_{1}$ simplifies our later proofs.

Next, we provide a few lemmas that prove specific types of matrices $A$ are strongly Schur-Horn continuous. These lemmas contribute to the final proof of our main theorem.

Lemma 3.6. Any irreducible symmetric matrix $A \in \mathbb{R}^{n \times n}$ is strongly Schur-Horn continuous.

Proof. Denote the eigendecomposition of $A$ as $A=Q \Lambda Q^{\top}$, where $Q$ is the eigenvectors of $A$ and $\Lambda$ is the diagonal eigenvalue matrix. The diagonal entries of $A$ is denoted as $d \in \mathbb{R}^{n}$. From $\|\widetilde{\Lambda}-\Lambda\|_{\mathrm{F}}=O(\varepsilon)$, we construct the perturbed matrix $B=Q \widetilde{\Lambda} Q^{\top}$ with $\|B-A\|_{\mathrm{F}}=O(\varepsilon)$. When $\varepsilon$ is sufficiently small, we know that the nonzero entries in $A$ remain nonzero in $B$ and are $\Theta(1)$ with respect to $\varepsilon$. Our proof starts from $B$ and constructs the desired $\widetilde{\sim}$ step by step based on Lemma 2.1. In the end, the eigenvalues of $\widetilde{B}$ are $\widetilde{\Lambda}$ and the diagonal entries of $\widetilde{B}$ are the same as that of A.

The underlying undirected graph of $A$ is a connected graph since $A$ is symmetric and irreducible, where the graph is built according to the non-zero pattern of $A$. We construct a spanning tree $T$ covering all vertices in the graph. Starting from a leaf vertex $v_{i}$ in $T$, assume the parent vertex of $v_{i}$ is $v_{j}$. Taking the $2 \times 2$ principal submatrix of $B$ at the intersections of the $i$-th and $j$-th columns and rows, we obtain a symmetric matrix as in Lemma 2.1,

$$
B^{(i j)}=\left[\begin{array}{ll}
B_{i i} & B_{i j} \\
B_{j i} & B_{j j}
\end{array}\right],
$$

where $B_{i i}$ is always in the $(1,1)$ entry of $B^{(i j)}$. The off-diagonal entries of $B^{(i j)}$ are nonzero and $\Theta(1)$ with respect to $\varepsilon$. By the first scenario in Lemma 2.1, there exists a Givens rotation matrix $G^{(i j)}$ such that the $(1,1)$ entry of $G^{(i j)} B^{(i j)}\left(G^{(i j)}\right)^{\top}$ is the target diagonal value $d_{i}$. Also, $G^{(i j)}$ is close to an identity matrix when $\varepsilon$ is sufficiently small, i.e., $\left\|G^{(i j)}-I\right\|_{\mathrm{F}}=\Theta\left(\varepsilon^{\alpha_{1}}\right)$ with $\alpha_{1} \geq 1$. Embed $G^{(i j)}$ into an $n \times n$ Givens rotation matrix $G_{1}$ with entries at the intersections of the $i$-th and $j$-th columns and rows. The $(i, i)$ entry of $G_{1} B G_{1}^{\top}$ is again $d_{i}$. Importantly, when $\varepsilon$ is sufficiently small, this operation only changes $(i, i)$ and $(j, j)$ entries along the diagonal of $B$ and preserves the connectivity in the graph and, hence, the spanning tree. ${ }^{5}$ After this step, the $i$-th diagonal entry has been "corrected" and all later operations will not touch it anymore. Hence, we can see that the vertex $v_{i}$ has been eliminated from the graph and tree. Then, we repeat this process with the updated tree and the matrix $G_{1} B G_{1}^{\top}$. Such a process could be repeated $n-1$ times and results $G_{1}, \ldots, G_{n-1}$ Givens rotation matrices. Note that the perturbation order of the diagonals is always $O(\varepsilon)$ during the procedure, we have $\left\|G_{i}-I\right\|_{\mathrm{F}}=\Theta\left(\varepsilon^{\alpha_{i}}\right)$ with $\alpha_{i} \geq 1$ for $i=1, \ldots, n-1$. Finally, we obtain a symmetric matrix,

$$
\widetilde{B}=G_{n-1} \cdots G_{1} B G_{1}^{\top} \cdots G_{n-1}^{\top}
$$

whose $n-1$ diagonal entries are "corrected" during the process and all the perturbations are collected at the last diagonal entry, which is automatically corrected due to the last majorization relation. Eigenvalues of $\widetilde{B}$ remain the same as $B$ during the similarity transformations. The difference between $\widetilde{B}$ and $A$ could be bounded as,

$$
\|\widetilde{B}-A\|_{\mathrm{F}} \leq\|\widetilde{B}-B\|_{\mathrm{F}}+\|B-A\|_{\mathrm{F}}=O(\varepsilon)
$$

Thus, $A$ is strongly Schur-Horn continuous.
Lemma 3.7. Given $A_{1} \in \mathbb{R}^{n_{1} \times n_{1}}$ and $A_{2} \in \mathbb{R}^{n_{2} \times n_{2}}$ be two strongly Schur-Horn continuous matrices such that their spectrum windows have a nonzero measured in-

[^5]tersection, i.e.,
$$
\mu\left(\omega\left(A_{1}\right) \cap \omega\left(A_{2}\right)\right)>0,
$$

where $\mu(\cdot)$ is the Lebesgue measure over $\mathbb{R}$, then matrix $A=\left[\begin{array}{ll}A_{1} & \\ & A_{2}\end{array}\right]$ is also strongly Schur-Horn continuous.

Proof. Denote the eigendecomposition of $A_{i}$ as $A_{i}=Q_{i} \Lambda_{i} Q_{i}^{\top}$ for $i=1,2$, where $Q_{i}$ is the eigenvector matrix and $\Lambda_{i}$ is the diagonal eigenvalue matrix, respectively. Given a perturbation of eigenvalues of $A$, denoted as $\widetilde{\Lambda}=\left[\begin{array}{ll}\widetilde{\Lambda}_{1} & \\ & \widetilde{\Lambda}_{2}\end{array}\right]$ with $\operatorname{tr}\left(\widetilde{\Lambda}_{1}\right)=$ $\operatorname{tr}\left(\Lambda_{1}\right)+h(\varepsilon)$ and $\operatorname{tr}\left(\widetilde{\Lambda}_{2}\right)=\operatorname{tr}\left(\Lambda_{2}\right)-h(\varepsilon)$. Since $\|\Lambda-\widetilde{\Lambda}\|_{\mathrm{F}}=O(\varepsilon)$, if $h(\varepsilon) \neq 0$, we must have $h(\varepsilon)=O(\varepsilon)=\Theta\left(\varepsilon^{\alpha}\right)$ with $\alpha \geq 1$.

Without loss of generality, we assume that $\lambda_{\min }\left(A_{1}\right) \leq \lambda_{\min }\left(A_{2}\right)$. By the assumption $\mu\left(\omega\left(A_{1}\right) \cap \omega\left(A_{2}\right)\right)>0$, spectrum windows of $A_{1}$ and $A_{2}$ admit either of the following cases,

$$
\lambda_{\min }\left(A_{1}\right) \leq \lambda_{\min }\left(A_{2}\right)<\lambda_{\max }\left(A_{1}\right) \leq \lambda_{\max }\left(A_{2}\right)
$$

or

$$
\lambda_{\min }\left(A_{1}\right) \leq \lambda_{\min }\left(A_{2}\right)<\lambda_{\max }\left(A_{2}\right) \leq \lambda_{\max }\left(A_{1}\right)
$$

In both cases, we have

$$
\lambda_{\min }\left(A_{1}\right)<\lambda_{\max }\left(A_{2}\right) \quad \text { and } \quad \lambda_{\min }\left(A_{2}\right)<\lambda_{\max }\left(A_{1}\right)
$$

and hence,

$$
\begin{equation*}
\lambda_{\min }\left(\widetilde{\Lambda}_{1}\right)<\lambda_{\max }\left(\widetilde{\Lambda}_{2}\right) \quad \text { and } \quad \lambda_{\min }\left(\widetilde{\Lambda}_{2}\right)<\lambda_{\max }\left(\widetilde{\Lambda}_{1}\right) \tag{3.2}
\end{equation*}
$$

when $\varepsilon$ is sufficiently small. Next, we split the discussion based on the sign of $h(\varepsilon)$.
If $h(\varepsilon)>0$, by Lemma 2.1, one can apply a Givens rotation between $\lambda_{\max }\left(\widetilde{\Lambda}_{1}\right)$ and $\lambda_{\min }\left(\widetilde{\Lambda}_{2}\right)$ to compensate $-h(\varepsilon)$,

$$
\left[\begin{array}{cc}
\lambda_{\max }\left(\widetilde{\Lambda}_{1}\right) & 0 \\
0 & \lambda_{\min }\left(\widetilde{\Lambda}_{2}\right)
\end{array}\right] \rightarrow\left[\begin{array}{cc}
\lambda_{\max }\left(\widetilde{\Lambda}_{1}\right)-h(\varepsilon) & \\
* & \lambda_{\min }\left(\widetilde{\Lambda}_{2}\right)+h(\varepsilon)
\end{array}\right]
$$

which is the second scenario in Table 1. The rotation is a perturbation of an identity whose rotation angle $\theta=\Theta\left(\varepsilon^{\alpha / 2}\right)=O\left(\varepsilon^{1 / 2}\right)$ since $\alpha \geq 1$. We embed the 2-by- 2 rotation matrix into a rotation matrix $G_{0}$ of the same size as $A$, which is also a perturbation of an identity and $\left\|G_{0}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$.

If $h(\varepsilon)<0$, by Lemma 2.1, one can apply a Givens rotation between $\lambda_{\min }\left(\widetilde{\Lambda}_{1}\right)$ and $\lambda_{\max }\left(\widetilde{\Lambda}_{2}\right)$ to compensate $-h(\varepsilon)$,

$$
\left[\begin{array}{cc}
\lambda_{\min }\left(\widetilde{\Lambda}_{1}\right) & 0 \\
0 & \lambda_{\max }\left(\widetilde{\Lambda}_{2}\right)
\end{array}\right] \rightarrow\left[\begin{array}{cc}
\lambda_{\min }\left(\widetilde{\Lambda}_{1}\right)-h(\varepsilon) & \\
* & \lambda_{\max }\left(\widetilde{\Lambda}_{2}\right)+h(\varepsilon)
\end{array}\right]
$$

which is the second scenario in Table 1. The rotation is a perturbation of an identity whose rotation angle $\theta=\Theta\left(\varepsilon^{\alpha / 2}\right)=O\left(\varepsilon^{1 / 2}\right)$ since $\alpha \geq 1$. We embed the 2 -by- 2
rotation matrix into a rotation matrix $G_{0}$ of the same size as $A$, which is also a perturbation of an identity and $\left\|G_{0}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$.

If $h(\varepsilon)=0$, we simply set $G_{0}=I$ and hence $\left\|G_{0}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$.
After the diagonal compensation, we denote the block form of $G_{0} \widetilde{\Lambda} G_{0}^{\top}$ as,

$$
G_{0} \widetilde{\Lambda} G_{0}^{\top}=\left[\begin{array}{cc}
\Lambda_{1}^{\prime} & E_{12} \\
E_{21} & \Lambda_{2}^{\prime}
\end{array}\right]
$$

For diagonal blocks, $\Lambda_{1}^{\prime}$ and $\Lambda_{2}^{\prime}$ are diagonal matrices being $O(\varepsilon)$ perturbation of $\Lambda_{1}$ and $\Lambda_{2}$ respectively, satisfying $\operatorname{tr}\left(\Lambda_{1}^{\prime}\right)=\operatorname{tr}\left(\Lambda_{1}\right)=\operatorname{tr}\left(A_{1}\right)$ and $\operatorname{tr}\left(\Lambda_{2}^{\prime}\right)=\operatorname{tr}\left(\Lambda_{2}\right)=$ $\operatorname{tr}\left(A_{2}\right)$. Off-diagonal blocks $E_{12}$ and $E_{21}$ are $O\left(\varepsilon^{1 / 2}\right)$ perturbations of zero matrices.

Since $A_{1}$ and $A_{2}$ are strongly Schur-Horn continuous, there exists

$$
B_{1}=G_{12} Q_{1} G_{11} \Lambda_{1}^{\prime} G_{11}^{\top} Q_{1}^{\top} G_{12}^{\top} \quad \text { and } \quad B_{2}=G_{22} Q_{2} G_{21} \Lambda_{2}^{\prime} G_{21}^{\top} Q_{2}^{\top} G_{22}^{\top}
$$

such that $\operatorname{diag}\left(B_{i}\right)=\operatorname{diag}\left(A_{i}\right)$ and $G_{i j}$ s being $O\left(\varepsilon^{1 / 2}\right)$ perturbation of identity matrices with $\left\|G_{i j}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i, j=1,2$. Introducing

$$
G_{1}=\left[\begin{array}{ll}
G_{11} & \\
& G_{21}
\end{array}\right] G_{0}, \quad Q=\left[\begin{array}{ll}
Q_{1} & \\
& Q_{2}
\end{array}\right], \quad G_{2}=\left[\begin{array}{ll}
G_{12} & \\
& G_{22}
\end{array}\right]
$$

we obtain

$$
\widetilde{B}=G_{2} Q G_{1} \widetilde{\Lambda} G_{1}^{\top} Q^{\top} G_{2}^{\top}
$$

which has eigenvalues being $\widetilde{\Lambda}$ and diagonal entries being the same as that of $A$. Since all $G_{i j} \mathrm{~s}$ and $G_{0}$ are orthogonal matrices close to identity matrices in order $O\left(\varepsilon^{1 / 2}\right)$ under the Frobenius norm, $G_{1}$ and $G_{2}$ are also orthogonal matrices being $O\left(\varepsilon^{1 / 2}\right)$ perturbations of identity matrices, i.e., $\left\|G_{i}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i=1,2$. Hence, we conclude that $A$ is strongly Schur-Horn continuous.

Lemma 3.8. Let $A_{1} \in \mathbb{R}^{n \times n}$ be a strongly Schur-Horn continuous matrix with spectrum window such that $\mu\left(\omega\left(A_{1}\right)\right)>0$, then for any $d_{2} \in \omega\left(A_{1}\right)^{\circ}$, matrix $A=$ $\left[\begin{array}{ll}A_{1} & \\ & d_{2}\end{array}\right]$ is strongly Schur-Horn continuous.

The proof of Lemma 3.8 is similar to that of Lemma 3.7. Note that from $d_{2} \in$ $\omega\left(A_{1}\right)^{\circ}$ we have $\lambda_{\min }\left(A_{1}\right)<d_{2}<\lambda_{\max }\left(A_{1}\right)$, which is the analogy of (3.2) in the proof of Lemma 3.7. Once we obtain these inequalities, the rest of the proofs are identical. Hence, we omit the detail for simplicity.

Lemma 3.9. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. There exists a permutation matrix $P$ such that the permuted matrix $P A P^{\top}$ admits a block diagonal structure,

$$
P A P^{\top}=\left[\begin{array}{lll}
A_{1} & & \\
& \ddots & \\
& & A_{p}
\end{array}\right]
$$

and diagonal blocks $\left\{A_{i}\right\}_{i=1}^{p}$ are either scalars or non-scalar strongly Schur-Horn continuous matrices. Their eigenvalues are ordered, i.e.,

$$
\begin{equation*}
\lambda_{\min }\left(A_{1}\right) \leq \lambda_{\max }\left(A_{1}\right) \leq \lambda_{\min }\left(A_{2}\right) \leq \lambda_{\max }\left(A_{2}\right) \leq \cdots \leq \lambda_{\min }\left(A_{p}\right) \leq \lambda_{\max }\left(A_{p}\right) \tag{3.3}
\end{equation*}
$$

Proof. Given a symmetric matrix $A$, let $q$ be the number of irreducible submatrices in $A$ (including scalars). We sort these submatrices in ascending order with their smallest eigenvalue being the first index and their largest eigenvalues being the second index. Hence, there exists a permutation matrix $P$ such that,

$$
P A P^{\top}=\left[\begin{array}{lll}
B_{1} & & \\
& \ddots & \\
& & B_{q}
\end{array}\right]
$$

and the smallest and largest eigenvalues of $i$-th and $j$-th blocks for $i<j$ admit either,
(i) $\lambda_{\text {min }}\left(B_{i}\right)<\lambda_{\text {min }}\left(B_{j}\right)$; or
(ii) $\lambda_{\min }\left(B_{i}\right)=\lambda_{\min }\left(B_{j}\right)$ and $\lambda_{\max }\left(B_{i}\right) \leq \lambda_{\max }\left(B_{j}\right)$.

Following these two eigenvalue conditions, we obtain the inequality immediately,

$$
\lambda_{\min }\left(\operatorname{diag}\left(B_{i}, B_{i+1}, \ldots, B_{q}\right)\right)=\lambda_{\min }\left(B_{i}\right)
$$

Next, we describe a procedure to collect contiguous $B_{i}$ s and denote them as $A_{j}$ such that the smallest and largest eigenvalues of $A_{j} \mathrm{~s}$ are ordered.

We construct $A_{1}$ step-by-step and start with $A_{1}=B_{1}$. By the construction of $P A P^{\top}$, we know that $B_{1}$ is either a diagonal matrix or an irreducible matrix. If $B_{1}$ is an irreducible matrix, then by Lemma 3.6, $B_{1}$ is strongly Schur-Horn continuous. Hence, $A_{1}=B_{1}$ is either a scalar or a strongly Schur-Horn continuous matrix.

We first consider the case that $A_{1}$ is a scalar and $A_{1}$ by itself is a block. Combined with the construction of $B_{i} \mathrm{~s}$, we have,

$$
\begin{equation*}
\lambda_{\max }\left(A_{1}\right)=B_{1} \leq \lambda_{\min }\left(B_{2}\right)=\lambda_{\min }\left(\operatorname{diag}\left(B_{2}, \ldots, B_{q}\right)\right) \tag{3.4}
\end{equation*}
$$

The second case is that $A_{1}$ is a strongly Schur-Horn continuous matrix. Note that according to Lemma B.1, irreducible matrix $B_{1}$ satisfies that $\mu\left(\omega\left(B_{1}\right)\right)>0$. In this case, we expand $A_{1}=\operatorname{diag}\left(B_{1}, \ldots, B_{i}\right)$ only if either $B_{i}$ is an irreducible matrix and $\mu\left(\omega\left(A_{1}\right) \cap \omega\left(B_{i}\right)\right)>0$; or $B_{i}$ is a scalar and $B_{i} \in \omega\left(A_{1}\right)^{\circ}$. By Lemma 3.7 and Lemma 3.8 respectively, the expanded matrix $A_{1}=\operatorname{diag}\left(B_{1}, \ldots, B_{i}\right)$ is strongly Schur-Horn continuous. If the expansion terminates, we have $\mu\left(\omega\left(A_{1}\right) \cap \omega\left(B_{i}\right)\right)=0$ when $B_{i}$ is an irreducible matrix, and $\lambda_{\max }\left(A_{1}\right) \leq B_{i}$ when $B_{i}$ is a scalar. Combined with the construction of $A_{1}$ and $B_{i} \mathrm{~s}$, we have,

$$
\begin{equation*}
\lambda_{\max }\left(A_{1}\right) \leq \lambda_{\min }\left(B_{i}\right)=\lambda_{\min }\left(\operatorname{diag}\left(B_{i}, \ldots, B_{q}\right)\right) \tag{3.5}
\end{equation*}
$$

with $A_{1}=\operatorname{diag}\left(B_{1}, \ldots, B_{i-1}\right)$ and $i>1$.
When the expansion of $A_{1}$ terminates at block $B_{i}$, the construction of $A_{2}$ starts from $B_{i}$ following the same procedure as above. Eventually, we obtain $A_{1}, \ldots, A_{p}$ and the inequality (3.3) follows from (3.4) and (3.5) directly.
4. Schur-Horn Continuity of Symmetric Matrices. Now we turn to the proof of our main result, Theorem 1.3.

Proof. Our proof is similar to that of Theorem 2.2, where diagonal scalars are replaced by either scalars or strongly Schur-Horn continuous blocks.

By Lemma 3.9, the symmetric matrix $A$ admits a block diagonal form after a permutation. Without loss of generality, we assume $A$ is a block diagonal matrix, $A=\operatorname{diag}\left(A_{1}, \cdots, A_{p}\right)$, where $A_{i}$ is either a scalar or a strongly Schur-Horn continuous matrix. The smallest and largest eigenvalues of these blocks are ordered as in
(3.3). The majorization conditions hold for every symmetric block $A_{i}$ and we have $\lambda_{\min }\left(A_{i}\right) \leq d_{\min }\left(A_{i}\right)$ and $d_{\max }\left(A_{i}\right) \leq \lambda_{\max }\left(A_{i}\right) .{ }^{6}$ Combined with (3.3), we have

$$
d_{\min }\left(A_{1}\right) \leq d_{\max }\left(A_{1}\right) \leq d_{\min }\left(A_{2}\right) \leq d_{\max }\left(A_{2}\right) \leq \cdots \leq d_{\min }\left(A_{p}\right) \leq d_{\max }\left(A_{p}\right)
$$

The perturbed eigenvalues are denoted as $\tilde{\lambda}_{i}$. When the eigenvalues have a gap, i.e., $\lambda_{i}<\lambda_{i+1}$, the perturbed eigenvalues keep the ordering, i.e., $\tilde{\lambda}_{i} \leq \tilde{\lambda}_{i+1}$ for sufficiently small $\varepsilon$. When eigenvalues are identical, $\lambda_{i}=\lambda_{i+1}$, the perturbed eigenvalues are ordered based on their perturbations. In another point of view, we could regard the perturbations on identical eigenvalues as sorted perturbations. Therefore, for $\varepsilon>0$ sufficiently small, we could obtain the block-wise majorization conditions,

$$
\begin{aligned}
\operatorname{tr}\left(\widetilde{\Lambda}_{1}\right) & \leq \operatorname{tr}\left(A_{1}\right) \\
\operatorname{tr}\left(\widetilde{\Lambda}_{1}\right)+\operatorname{tr}\left(\widetilde{\Lambda}_{2}\right) & \leq \operatorname{tr}\left(A_{1}\right)+\operatorname{tr}\left(A_{2}\right) \\
& \vdots \\
\operatorname{tr}\left(\widetilde{\Lambda}_{1}\right)+\cdots+\operatorname{tr}\left(\widetilde{\Lambda}_{p-1}\right) & \leq \operatorname{tr}\left(A_{1}\right)+\cdots+\operatorname{tr}\left(A_{p-1}\right) \\
\operatorname{tr}\left(\widetilde{\Lambda}_{1}\right)+\cdots+\operatorname{tr}\left(\widetilde{\Lambda}_{p-1}\right)+\operatorname{tr}\left(\widetilde{\Lambda}_{p}\right) & =\operatorname{tr}\left(A_{1}\right)+\cdots+\operatorname{tr}\left(A_{p-1}\right)+\operatorname{tr}\left(A_{p}\right),
\end{aligned}
$$

where $\widetilde{\Lambda}_{1}, \ldots, \widetilde{\Lambda}_{p}$ are perturbed diagonal eigenvalue submatrices corresponding to the block structure as in $A$. We denote the block-wise perturbations as $h_{i}(\varepsilon)=\operatorname{tr}\left(\widetilde{\Lambda}_{i}\right)-$ $\operatorname{tr}\left(A_{i}\right)$, and similarly have,

$$
\begin{aligned}
& h_{1}(\varepsilon)+h_{2}(\varepsilon)+\cdots+h_{i}(\varepsilon) \leq 0, \quad i=1,2, \ldots, p-1, \text { and } \\
& h_{1}(\varepsilon)+\cdots+h_{p}(\varepsilon)=0
\end{aligned}
$$

We also maintain a priority queue with diagonal block indices as elements. For any diagonal block index $i$ in the queue, we ensure that $\tilde{h}_{i}(\varepsilon)$ is a negative perturbation $\tilde{h}_{i}(\varepsilon)<0$, where $\tilde{h}_{i}(\varepsilon)$ denotes the updated perturbation throughout the procedure. Starting from the first diagonal block, we check and enqueue the index $i=1,2, \ldots$ in order if $\tilde{h}_{i}(\varepsilon)<0$ and skip the index $i$ if $\tilde{h}_{i}(\varepsilon)=0$. We keep on checking and enqueuing indices until the first index $j$ such that $\breve{h}_{j}(\varepsilon)>0$. If $j$ does not exist, then by the last equation in the block-wise majorization relation, we know that the queue is also empty and the diagonal blocks of the perturbed $A$ have all been corrected, i.e.,

$$
\begin{equation*}
\tilde{h}_{i}(\varepsilon)=0, \quad i=1, \ldots, p \tag{4.1}
\end{equation*}
$$

Otherwise, we obtain a $j$ and the updated perturbations satisfy,

$$
\begin{equation*}
\tilde{h}_{1}(\varepsilon)+\cdots+\tilde{h}_{j}(\varepsilon)=h_{1}(\varepsilon)+\cdots+h_{j}(\varepsilon) \leq 0 \tag{4.2}
\end{equation*}
$$

This condition is satisfied in the first step and we will verify it after each step. By the $j$-th block-wise majorization relation, the queue is guaranteed to be non-empty. We pop an index from the queue and denote it as $i$. There are two scenarios here based on the block size of $A_{i}$ and $A_{j}$.
(i) Both $A_{i}$ and $A_{j}$ are one dimensional, i.e., $A_{i}=d_{i}$ and $A_{j}=d_{j}$. This scenario is the same as Theorem 2.2 for diagonal matrix and we have (2.8). Thus, if $d_{i}<d_{j}$, the Givens rotation itself is close to identity; if $d_{i}=d_{j}$, while the Givens rotation could be away from identity, the rotated matrix is close to the original block diagonal matrix.

[^6](ii) At least one of $A_{i}$ and $A_{j}$ is non-diagonal strongly Schur-Horn continuous matrix. By Proposition 3.5, we have either $\lambda_{\min }\left(A_{i}\right)<\lambda_{\max }\left(A_{i}\right)$ or $\lambda_{\min }\left(A_{j}\right)<$ $\lambda_{\max }\left(A_{j}\right)$, or both. Combined with the ordering of eigenvalues, (3.3), the inequality $\lambda_{\min }\left(A_{i}\right)<\lambda_{\max }\left(A_{j}\right)$ holds. Thus, by Lemma 2.1, we can perform a Givens rotation close to the identity between $\lambda_{\min }\left(A_{i}\right)$ and $\lambda_{\max }\left(A_{j}\right)$ to revert $-\tilde{h}_{i}(\varepsilon)$, making the sum of the $i$-th block perturbations equals zero, i.e.,
\[

\left[$$
\begin{array}{cc}
\lambda_{\min }\left(\widetilde{\Lambda}_{i}\right) & 0 \\
0 & \lambda_{\max }\left(\widetilde{\Lambda}_{j}\right)
\end{array}
$$\right] \rightarrow\left[\begin{array}{cc}
\lambda_{\min }\left(\widetilde{\Lambda}_{i}\right)-\tilde{h}_{i}(\varepsilon) \& * <br>

* \& \lambda_{\max }\left(\widetilde{\Lambda}_{j}\right)+\tilde{h}_{i}(\varepsilon)
\end{array}\right]
\]

We now have three cases: i) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)<0$; ii) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)=0$; and iii) $\tilde{h}_{i}(\varepsilon)+\tilde{h}_{j}(\varepsilon)>0$. In case i), we enqueue $j$ and start checking the following indices. In case ii), we skip $j$ and start checking the indices after $j$. In case iii), we pop another index from the queue and repeat the correction procedure. In all cases, the updated total perturbations at the $i$-th block and the $j$-th block are 0 and $\tilde{h}_{i}(\varepsilon)+$ $\tilde{h}_{j}(\varepsilon)$, respectively. Hence (4.2) holds for all indices greater or equal to $j$. Then, the majorization relations guarantee that the procedure ends if and only if all diagonal blocks have been corrected, i.e., equations (4.1) hold. Notice that all above corrections are on the diagonal eigenvalue matrix $\widetilde{\Lambda}$. The corrected matrix $\widetilde{\Lambda}^{(p)}$ admits form,

$$
\widetilde{\Lambda}^{(p)}=\left[\begin{array}{ccc}
\widetilde{\Lambda}_{1}^{(p)} & * & * \\
* & \ddots & * \\
* & * & \widetilde{\Lambda}_{p}^{(p)}
\end{array}\right]
$$

with $\widetilde{\Lambda}_{i}^{(p)}$ being diagonal matrices satisfying $\operatorname{tr}\left(\widetilde{\Lambda}_{i}^{(p)}\right)=\operatorname{tr}\left(A_{i}\right)$ for $i=1, \ldots, p$. The diagonals of $\widetilde{\Lambda}^{(p)}$ are always $O(\varepsilon)$ perturbation to diagonals of $\Lambda$ during the trace adjustment between diagonal blocks, i.e.,

$$
\left\|\widetilde{\Lambda}_{i}^{(p)}-\Lambda_{i}\right\|_{\mathrm{F}}=O(\varepsilon), \quad i=1, \ldots, p
$$

Furthermore, analog to the proof of Theorem 2.2, it yields that all the off-diagonals of $\widetilde{\Lambda}^{(p)}$ are $O\left(\varepsilon^{1 / 2}\right)$. Thus, the distance between $\widetilde{\Lambda}$ and $\widetilde{\Lambda}^{(p)}$ obeys,

$$
\begin{equation*}
\left\|\widetilde{\Lambda}-\widetilde{\Lambda}^{(p)}\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right) \tag{4.3}
\end{equation*}
$$

If the $i$-th diagonal block is a scalar, the correction procedure guarantees the diagonal entry is correct. If the $i$-th diagonal block is a strongly Schur-Horn continuous matrix, then the Definition 3.2 ensures the existence of $G_{i 1}$ and $G_{i 2}$ close to identity and $\widetilde{B}_{i}=G_{i 2} Q_{i} G_{i 1} \widetilde{\Lambda}_{i}^{(p)} G_{i 1}^{\top} Q_{i}^{\top} G_{i 2}^{\top}$ has the same diagonal entries as $A_{i}$, where $Q_{i}$ is the eigenvector matrix of $A_{i}$. Assembling $\left\{Q_{i}\right\},\left\{G_{i 1}\right\}$, and $\left\{G_{i 2}\right\}$ together, we denote them as,

$$
G_{1}=\left[\begin{array}{lll}
G_{11} & & \\
& \ddots & \\
& & G_{p 1}
\end{array}\right], \quad Q=\left[\begin{array}{lll}
Q_{1} & & \\
& \ddots & \\
& & Q_{p}
\end{array}\right], \quad G_{2}=\left[\begin{array}{lll}
G_{12} & & \\
& \ddots & \\
& & G_{p 2}
\end{array}\right]
$$

where $G_{i 1}=Q_{i}=G_{i 2}=1$ if $A_{i}$ is a scalar. By construction, we know that $G_{1}$ and $G_{2}$ are close to the identity matrix with $\left\|G_{i}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i=1,2$. The matrix
$\underset{\sim}{\widetilde{B}}=G_{2} Q G_{1} \widetilde{\Lambda}^{(p)} G_{1}^{\top} Q^{\top} G_{2}^{\top}$ has the same diagonal entries as $A$ and eigenvalues being $\widetilde{\Lambda}$. We verify that $\widetilde{B}$ is close to $A$,

$$
\begin{aligned}
\|\widetilde{B}-A\|_{\mathrm{F}} & =\left\|G_{2} Q G_{1} \widetilde{\Lambda}^{(p)} G_{1}^{\top} Q^{\top} G_{2}^{\top}-Q \Lambda Q^{\top}\right\|_{\mathrm{F}} \\
& \leq\left\|Q G_{1} \widetilde{\Lambda}^{(p)} G_{1}^{\top} Q^{\top}-Q \Lambda Q^{\top}\right\|_{\mathrm{F}}+O\left(\varepsilon^{1 / 2}\right) \\
& =\left\|G_{1} \widetilde{\Lambda}^{(p)} G_{1}^{\top}-\Lambda\right\|_{\mathrm{F}}+O\left(\varepsilon^{1 / 2}\right) \\
& \leq\left\|\widetilde{\Lambda}^{(p)}-\Lambda\right\|_{\mathrm{F}}+O\left(\varepsilon^{1 / 2}\right) \\
& \leq\left\|\widetilde{\Lambda}^{(p)}-\widetilde{\Lambda}\right\|_{\mathrm{F}}+\|\widetilde{\Lambda}-\Lambda\|_{\mathrm{F}}+O\left(\varepsilon^{1 / 2}\right)=O\left(\varepsilon^{1 / 2}\right)
\end{aligned}
$$

where the first and second inequalities are due to the fact that $G_{1}$ and $G_{2}$ are close to the identity matrix with $\left\|G_{i}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i=1,2$, the second equality is due to the unitary invariant property of the Frobenius norm, the last inequality is due to the norm triangular inequality, and the last equality is due to the definition of $\widetilde{\Lambda}$ and (4.3).
5. Schur-Horn Continuity of Hermitian Matrices. Next, we discuss the Schur-Horn continuity of a Hermitian matrix, as described in Theorem 1.5. The proof follows a similar approach as that for symmetric matrices, with an extra step to generalize Lemma 2.1 to its complex counterpart.

Denote $\mathrm{i}=\sqrt{-1}$, we introduce the complex Givens rotation defined as

$$
G=\left[\begin{array}{cc}
\mathrm{e}^{\mathrm{i} \phi} \cos \theta & \mathrm{e}^{\mathrm{i} \psi} \sin \theta  \tag{5.1}\\
-\mathrm{e}^{-\mathrm{i} \psi} \sin \theta & \mathrm{e}^{-\mathrm{i} \phi} \cos \theta
\end{array}\right]
$$

where $\theta, \phi, \psi \in \mathbb{R}$.
LEMmA 5.1. Given $\varepsilon>0$ small enough and $d_{1}, d_{2} \in \mathbb{R}$. Let a Hermitian matrix $B$ of form,

$$
B=\left[\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{array}\right]=\left[\begin{array}{cc}
d_{1}-f(\varepsilon) & b_{12} \\
b_{12}^{*} & d_{2}+g(\varepsilon)
\end{array}\right]
$$

with $f(\varepsilon)=\Theta\left(\varepsilon^{\alpha}\right), g(\varepsilon)=\Theta\left(\varepsilon^{\beta}\right)$ for $\alpha, \beta>0$. Further, we assume that

$$
\left|b_{12}\right|^{2}+f(\varepsilon)\left(d_{2}-d_{1}+g(\varepsilon)\right) \geq 0
$$

Then there exists a complex Givens rotation $G$ with rotation angle $\theta=\Theta\left(\varepsilon^{\gamma}\right)$ and $\phi, \psi \in \mathbb{R}$ such that the $(1,1)$ entry of $\widetilde{B}=G B G^{*}$ is $\tilde{b}_{11}=d_{1}$ and $\|\widetilde{B}-B\|_{\mathrm{F}}=O\left(\varepsilon^{\delta}\right)$ where various scenarios of $\gamma$ and $\delta$ are provided in Table 2.

Proof. Note that $B$ is a Hermitian matrix whose diagonal entries are all real numbers, if $b_{12}=0$, it reduces to Lemma 2.1 and we have the conclusion for such scenarios directly. Below we assume that $b_{12} \neq 0$. Denote $b_{12}=\operatorname{Re}\left(b_{12}\right)+\mathrm{i} \operatorname{Im}\left(b_{12}\right)$ where $\operatorname{Re}\left(b_{12}\right) \in \mathbb{R}$ and $\operatorname{Im}\left(b_{12}\right) \in \mathbb{R}$ are real part and imaginary part of $b_{12}$, respectively. We further split the discussion into two scenarios: (i) $\operatorname{Re}\left(b_{12}\right) \neq 0$ and (ii) $\operatorname{Re}\left(b_{12}\right)=0$.

First, if $\operatorname{Re}\left(b_{12}\right) \neq 0$, we still apply the real Givens rotation denoted as $G=$

| Various Scenarios | $\gamma$ | $\delta$ |
| :---: | :---: | :---: |
| $b_{12} \neq 0$ | $\alpha$ | $\alpha$ |
| $b_{12}=0 \quad d_{1} \neq d_{2}$ | $\alpha / 2$ | $\alpha / 2$ |
| $b_{12}=0 \quad d_{1}=d_{2} \quad \alpha>\beta$ | $(\alpha-\beta) / 2$ | $(\alpha+\beta) / 2$ |
| $b_{12}=0 \quad d_{1}=d_{2} \quad \alpha \leq \beta$ | 0 | $\alpha$ |

Various scenarios of $b_{12}, d_{1}, d_{2}, \gamma$, and $\delta$ for Lemma 5.1.
$\left[\begin{array}{cc}c & s \\ -s & c\end{array}\right]$ with $c=\cos \theta$ and $s=\sin \theta$ to matrix $B$ and obtain,

$$
G B G^{*}=\left[\begin{array}{cc}
c^{2} b_{11}+s^{2} b_{22}+2 c s \operatorname{Re}\left(b_{12}\right) & \omega \\
\omega^{*} & c^{2} b_{22}+s^{2} b_{11}-2 c s \operatorname{Re}\left(b_{12}\right)
\end{array}\right]
$$

with $\omega=c s\left(b_{22}-b_{11}\right)+\left(c^{2}-s^{2}\right) \operatorname{Re}\left(b_{12}\right)+\mathrm{i} \operatorname{Im}\left(b_{12}\right)$. Equating the $(1,1)$ entry of $\widetilde{B}=G B G^{*}$ and $d_{1}$ it leads to

$$
c^{2} b_{11}+s^{2} b_{22}+2 c s \operatorname{Re}\left(b_{12}\right)=d_{1}
$$

which is analogous to (2.2) in the proof of Lemma 2.1 with $b_{12}$ replaced by $\operatorname{Re}\left(b_{12}\right)$. Note that $\operatorname{Re}\left(b_{12}\right) \neq 0$, adopting a similar analysis one concludes that $\gamma=\alpha$ and $\delta=\alpha$.

Second, if $\operatorname{Re}\left(b_{12}\right)=0$, at this time from $b_{12} \neq 0$ we must have $\operatorname{Im}\left(b_{12}\right) \neq 0$. Thus we consider another Givens rotation matrix denoted as $G=\left[\begin{array}{cc}\mathrm{i} c & s \\ -s & -\mathrm{i} c\end{array}\right]$, with $\phi=\frac{\pi}{2}$ and $\psi=0$. Applying $G$ from the left of $B$ and $G^{*}$ from the right of $B$ we get,

$$
G B G^{*}=\left[\begin{array}{cc}
c^{2} b_{11}+s^{2} b_{22}-2 c s \operatorname{Im}\left(b_{12}\right) & \omega \\
\omega^{*} & c^{2} b_{22}+s^{2} b_{11}+2 c s \operatorname{Im}\left(b_{12}\right)
\end{array}\right]
$$

with $\omega=\mathrm{i} c s\left(b_{22}-b_{11}\right)-\mathrm{i}\left(c^{2}-s^{2}\right) \operatorname{Im}\left(b_{12}\right)-\operatorname{Re}\left(b_{12}\right)$. Equating the $(1,1)$ entry of $\widetilde{B}=G B G^{*}$ and $d_{1}$ it leads to

$$
c^{2} b_{11}+s^{2} b_{22}-2 c s \operatorname{Im}\left(b_{12}\right)=d_{1}
$$

which is analogous to (2.2) in the proof of Lemma 2.1 with $b_{12}$ replaced by $-\operatorname{Im}\left(b_{12}\right)$. Note that $\operatorname{Im}\left(b_{12}\right) \neq 0$, adopting a similar analysis one concludes that $\gamma=\alpha$ and $\delta=\alpha$.

Now, we generalize the definition of strong Schur-Horn continuity for symmetric matrices to Hermitian matrices, by simply replacing orthogonal matrices in Definition 3.2 with unitary matrices.

Definition 5.2 (Strong Schur-Horn Continuity for Hermitian Matrices). Suppose $A \in \mathbb{C}^{n \times n}$ is a Hermitian matrix with an eigendecomposition $A=Q \Lambda Q^{*}$, where $Q$ is the unitary eigenvector matrix and $\Lambda$ is the diagonal eigenvalue matrix. Matrix $A$ is strongly Schur-Horn continuous if, for any perturbed eigenvalues $\widetilde{\Lambda}$ satisfying $\operatorname{tr}(\widetilde{\Lambda})=\operatorname{tr}(\Lambda)$ and $\|\widetilde{\Lambda}-\Lambda\|_{\mathrm{F}}=O(\varepsilon)$ for $\varepsilon>0$ sufficiently small, there exists a Hermitian matrix $\widetilde{B}=G_{2} Q G_{1} \widetilde{\Lambda} G_{1}^{*} Q^{*} G_{2}^{*}$ such that

1. $\operatorname{diag}(\widetilde{B})=\operatorname{diag}(A)$,
2. $G_{1}$ and $G_{2}$ are unitary matrices, and
3. $\left\|G_{i}-I\right\|_{\mathrm{F}}=O\left(\varepsilon^{1 / 2}\right)$ for $i=1,2$.

With Lemma 5.1, one can verify that the desired properties of strong Schur-Horn continuity appeared in section 3 also hold for Hermitian matrices. Thus, by employing a similar analysis, one can prove Theorem 1.5 and establish the Schur-Horn continuity of Hermitian matrices, and we omit the details.
6. Conclusion. In this paper, we explore the eigenvalue perturbation of a symmetric (Hermitian) matrix with fixed diagonals, which is referred to as the continuity of the Schur-Horn mapping. We first establish the Schur-Horn continuity for real diagonal matrices leveraging Givens rotation and majorization relations between diagonals and eigenvalues. Then, we introduce the concept of the strong Schur-Horn continuity, which is a stronger version of Schur-Horn continuity. This allows us to construct a block-wise majorization relation and prove the Schur-Horn continuity for general symmetric matrices. Additionally, our analysis could be extended to Hermitian matrices to establish their Schur-Horn continuity.

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## Appendix A. Proof of Proposition 3.5.

Proof. We prove by contrapositive. If $A$ is not a scalar matrix and does not satisfy (3.1), then we have either (i) $A$ not satisfying the majorization relation (1.1); or (ii) $A$ satisfying the majorization relation, but there exists $1 \leq i, j<n$ such that,

$$
\lambda_{1}+\cdots+\lambda_{i}=d_{1}+\cdots+d_{i}, \quad \text { and } \quad \lambda_{1}+\cdots+\lambda_{j}<d_{1}+\cdots+d_{j}
$$

In case (i), $A$ does not satisfy the majorization relation. Hence, by Schur-Horn theorem, $A$ is not strongly Schur-Horn continuous.

In case (ii), we have $\lambda_{1}<\lambda_{n}$. The discussion is further split into two scenarios: (ii.1) $\lambda_{i}<\lambda_{n}$ and (ii.2) $\lambda_{i}=\lambda_{n}$.

In (ii.1), we denote those eigenvalues equal to $\lambda_{i}$ and $\lambda_{n}$ as follows,

$$
\lambda_{\ell}<\lambda_{\ell+1}=\cdots=\lambda_{i}=\cdots=\lambda_{r}<\lambda_{r+1}, \quad \text { and } \quad \lambda_{k}<\lambda_{k+1}=\cdots=\lambda_{n}
$$

where $0 \leq \ell<i \leq r \leq k<n$. Then we consider a particular perturbation by adding an $\varepsilon>0$ small enough to those eigenvalues equal to $\lambda_{i}$ and subtracting $\frac{r-\ell}{n-k} \varepsilon$ from those eigenvalues equal to $\lambda_{n}$, i.e.,

$$
\begin{aligned}
\tilde{\lambda}=\left(\lambda_{1}, \cdots, \lambda_{\ell}, \lambda_{\ell+1}+\varepsilon, \cdots,\right. & \lambda_{i}+\varepsilon, \cdots, \\
& \lambda_{r}+\varepsilon \\
& \left.\lambda_{r+1}, \cdots, \lambda_{k}, \lambda_{k+1}-\frac{r-\ell}{n-k} \varepsilon, \cdots, \lambda_{n}-\frac{r-\ell}{n-k} \varepsilon\right)
\end{aligned}
$$

This perturbed $\widetilde{\Lambda}=\operatorname{diag}(\tilde{\lambda})$ satisfies $\operatorname{tr}(\widetilde{\Lambda})=\operatorname{tr}(\Lambda)$ and $\|\widetilde{\Lambda}-\Lambda\|_{\mathrm{F}}=O(\varepsilon)$. Furthermore, $\tilde{\lambda}$ is still in a non-decreasing order for $\varepsilon>0$ sufficiently small. While, the $i$-th majorization relation between $\tilde{\lambda}$ and $d$ is violated,

$$
\tilde{\lambda}_{1}+\cdots+\tilde{\lambda}_{i}=\lambda_{1}+\cdots+\lambda_{i}+(i-\ell) \varepsilon=d_{1}+\cdots+d_{i}+(i-\ell) \varepsilon>d_{1}+\cdots+d_{i}
$$

By Schur-Horn theorem, there does not exist a matrix with diagonal and eigenvalues being $d$ and $\tilde{\lambda}$, respectively. Hence, $A$ is not strongly Schur-Horn continuous.

In (ii.2), we have $\lambda_{1}<\lambda_{i}=\lambda_{n}$. Denote those eigenvalues equal to $\lambda_{1}$ and $\lambda_{i}$ as follows,

$$
\lambda_{1}=\cdots=\lambda_{r}<\lambda_{r+1} \quad \text { and } \quad \lambda_{k}<\lambda_{k+1}=\cdots=\lambda_{i}=\cdots=\lambda_{n}
$$

where $1 \leq r \leq k<n$. Consider a particular perturbation by subtracting $\varepsilon>0$ from those eigenvalues equal to $\lambda_{i}$ and adding $\frac{n-k}{r} \varepsilon$ to those eigenvalues equal to $\lambda_{1}$ as follows,

$$
\tilde{\lambda}=\left(\lambda_{1}+\frac{n-k}{r} \varepsilon, \cdots, \lambda_{r}+\frac{n-k}{r} \varepsilon, \lambda_{r+1}, \cdots, \lambda_{k}, \lambda_{k+1}-\varepsilon, \cdots, \lambda_{n}-\varepsilon\right) .
$$

This perturbed $\widetilde{\Lambda}=\operatorname{diag}(\tilde{\lambda})$ satisfies $\operatorname{tr}(\widetilde{\Lambda})=\operatorname{tr}(\Lambda)$ and $\|\widetilde{\Lambda}-\Lambda\|_{\mathrm{F}}=O(\varepsilon)$. Furthermore, $\tilde{\lambda}$ is in a non-decreasing order when $\varepsilon>0$ is sufficiently small. Similarly, the $i$-th majorization relation between $\tilde{\lambda}$ and $d$ is violated,

$$
\begin{aligned}
\tilde{\lambda}_{1}+\cdots+\tilde{\lambda}_{i}=\lambda_{1}+\cdots+\lambda_{i}+r \cdot \frac{n-k}{r} & \varepsilon-(i-k) \varepsilon \\
& =d_{1}+\cdots+d_{i}+(n-i) \varepsilon>d_{1}+\cdots+d_{i} .
\end{aligned}
$$

By Schur-Horn theorem, there does not exist a matrix with diagonal and eigenvalues being $d$ and $\tilde{\lambda}$, respectively. Hence, $A$ is not strongly Schur-Horn continuous.

## Appendix B. Spectrum Window for Irreducible Matrices.

Lemma B.1. If $A$ is an irreducible symmetric matrix whose dimension is strictly greater than 1, then $\lambda_{\min }(A)<\lambda_{\max }(A)$.

Proof. Consider the eigendecomposition $A=Q \Lambda Q^{\top}$ of the symmetric matrix $A$, where $Q$ is the orthonormal matrix composed of the eigenvectors of $A$, and $\Lambda$ is the diagonal eigenvalue matrix. Suppose $\lambda_{\min }(A)=\lambda_{\max }(A)=\lambda$, then we have $A=Q \cdot \lambda I \cdot Q^{\top}=\lambda I$, which contradicts the irreducibility of $A$.

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[^1]:    ${ }^{1}$ For the sake of notation, we adopt $\operatorname{diag}(\cdot)$ similar to the MATLAB "diag" function, i.e., $\operatorname{diag}(v)$ is a square diagonal matrix with the entries of vector $v$ on the diagonal and $\operatorname{diag}(A)$ is a column vector of the diagonal entries of $A$.

[^2]:    ${ }^{2}$ If the index $i$ is skipped, we assign $\widetilde{A}^{(i)}=\widetilde{A}^{(i-1)}$.

[^3]:    ${ }^{3}$ We dropped the superscript $(i-1)$ for simplicity.

[^4]:    ${ }^{4}$ We use $\lambda_{\min }(\cdot)$ and $\lambda_{\max }(\cdot)$ to denote the minimum and maximum eigenvalues of a matrix, respectively.

[^5]:    ${ }^{5}$ New edges could be added to the graph. However, the spanning tree is still a spanning tree in the updated graph.

[^6]:    ${ }^{6}$ Notations $d_{\min }(\cdot)$ and $d_{\max }(\cdot)$ denote the minimum and maximum of diagonal entries of a matrix, respectively.

