Rational Krylov Methods for Solving Nonlinear Eigenvalue Problems

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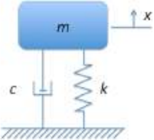
Quadratic eigenvalue problem



Vibration analysis in structural analysis gives rise to

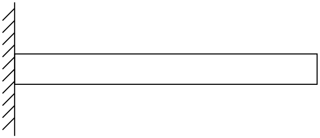
$$(\lambda^2 M + \lambda C + K)x = 0$$

where

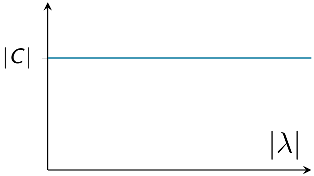
- λ is an eigenvalue
- x is an eigenvector
- M is the mass matrix
- C is the damping matrix
- K is the stiffness matrix

Motivation: Nonlinear damping

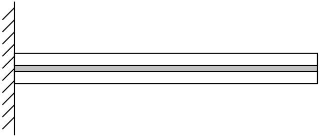
Clamped beam:



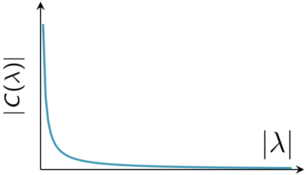
$$\left(\lambda^2 M + \lambda C + K\right) x = 0$$



Clamped sandwich beam:



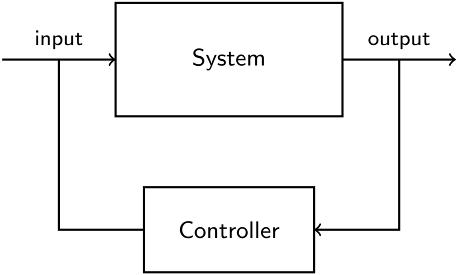
$$\left(\lambda^2 M + \lambda C(\lambda) + K\right) x = 0$$



for λ on the imaginary axis

Motivation: Active damping

Active damping in cars:



Delay eigenvalue problem

$$\left(\lambda^2 M + \lambda C + K + e^{-\lambda\tau} E\right) x = 0$$

Motivation: Nonlinear boundary conditions

Cavity design of a linear accelerator

$$\nabla \times \left(\frac{1}{\mu} \nabla \times E \right) - \lambda \varepsilon E = 0$$

Maxwell's equations + nonlinear waveguide boundary conditions:

$$\left(K - \lambda M + \sum_{j=1}^{k} i \sqrt{\lambda - \kappa_{c,j}^2} W_j \right) x = 0$$

where

- $\lambda = \omega^2 / c^2$
- $\kappa_{c,j}$ are the cutoff values

Outline

1. Motivation
2. Solving Nonlinear Eigenvalue Problems
   - Approximation
   - Linearization pencils
   - Solving linear eigenvalue problem
3. Numerical Experiment

Nonlinear eigenvalue problem (NLEP)

NLEP

The nonlinear eigenvalue problem:

$$A(\lambda)x = 0$$

where

- $\lambda \in \Omega \subseteq \mathbb{C}$: eigenvalue
- $x \in \mathbb{C}^n \backslash \{0\}$: eigenvector
- $A : \Omega \rightarrow \mathbb{C}^{n \times n}$: matrix-valued function

Nonlinear eigenvalue problem (NLEP)

NLEP

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Note that the NLEP is

$\rightsquigarrow$ nonlinear in eigenvalue λ,

$\rightsquigarrow$ linear in eigenvector x.

Solving NLEPs

NLEP

$A(\lambda)x = 0$

$\Downarrow$ 1. approximation via interpolation

PEP

$P_d(\lambda)x = 0$

$\Downarrow$ 2. linearization

GEP

$\mathbf{L}(\lambda)\mathbf{x} = 0$

$\Downarrow$ 3. solving linear eigenvalue problem

Solution

Approximation

NLEP

$A(\lambda)x = 0$

$\Downarrow$

PEP

$P_d(\lambda)x = 0$

$\Downarrow$

GEP

$\mathbf{L}(\lambda)\mathbf{x} = 0$

$\Downarrow$

Solution

Step 1: Polynomial interpolation

$$A(\lambda) \approx P_d(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_d\lambda^d$$

Approximation

NLEP

$A(\lambda)x = 0$

$\Downarrow$

PEP

$P_d(\lambda)x = 0$

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$\mathbf{L}(\lambda)\mathbf{x} = 0$

$\Downarrow$

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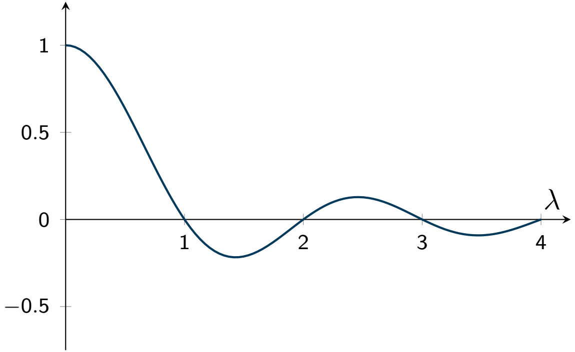

Approximation

NLEP

$A(\lambda)x = 0$

$\Downarrow$

PEP

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GEP

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$\Downarrow$

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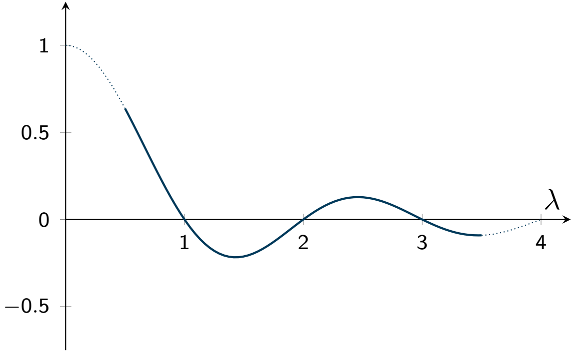

Approximation

NLEP

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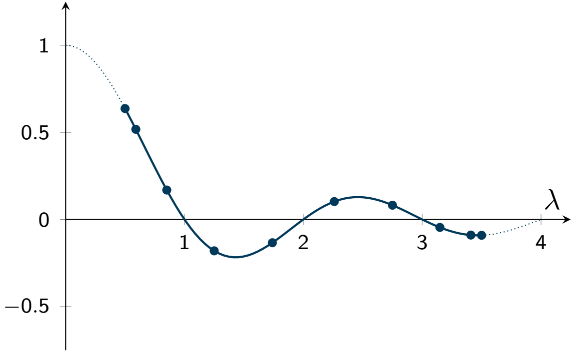

Approximation

NLEP

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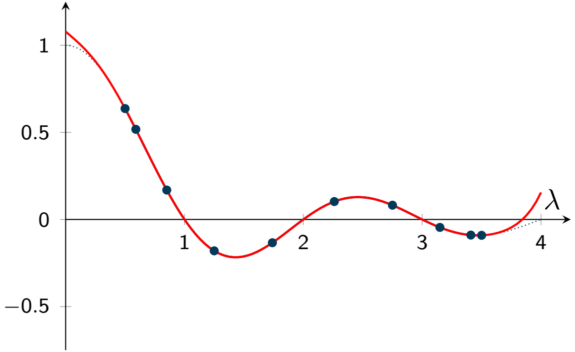
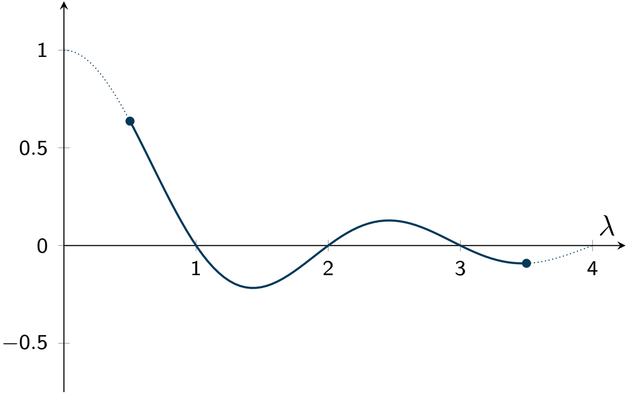


Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda)$$



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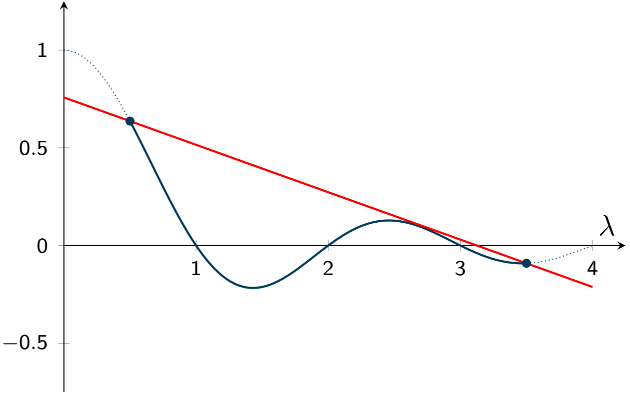
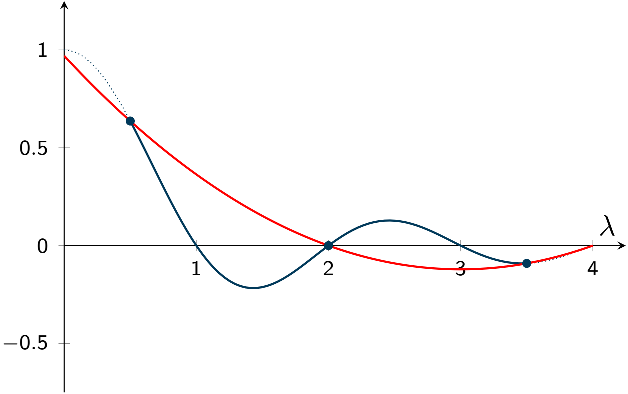


Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

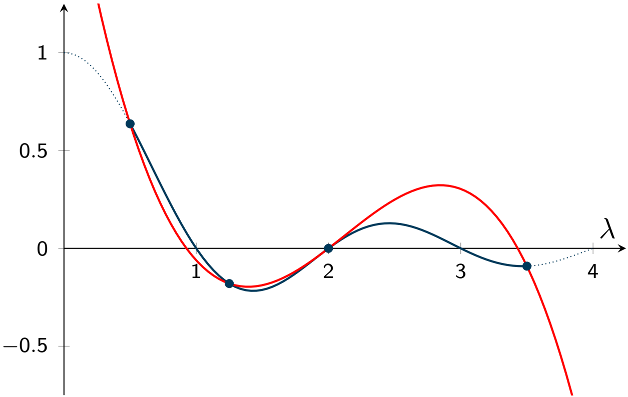
$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + A_2 n_2(\lambda)$$



Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

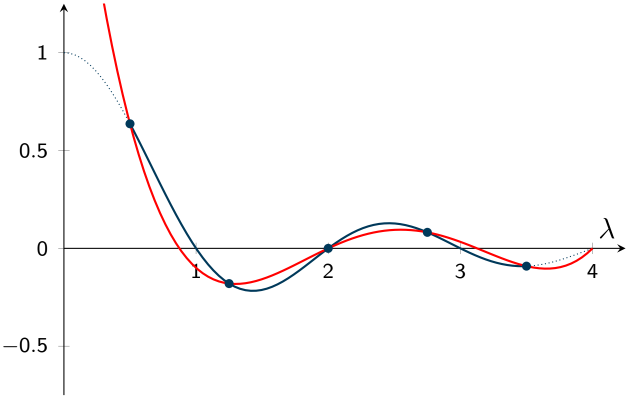
$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_3 n_3(\lambda)$$



Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_4 n_4(\lambda)$$



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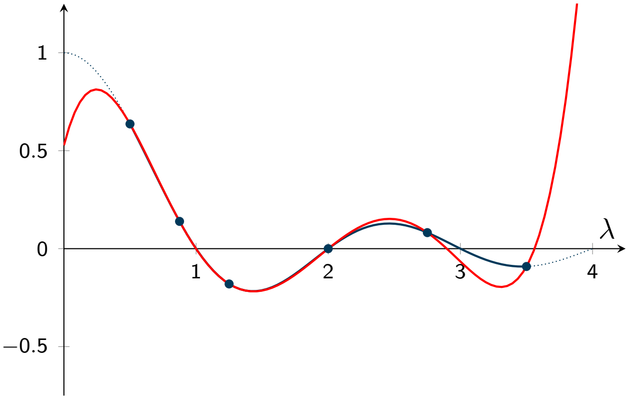
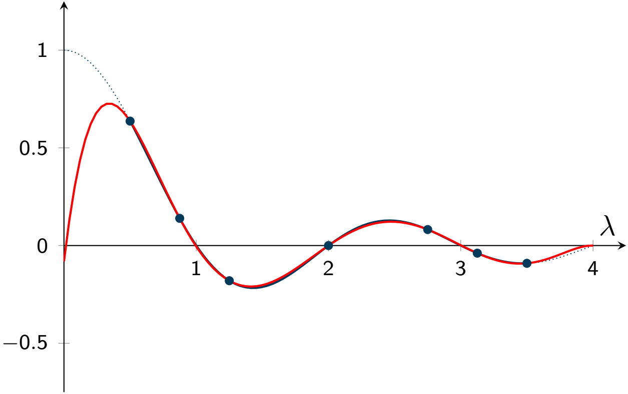
$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_5 n_5(\lambda)$$



Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

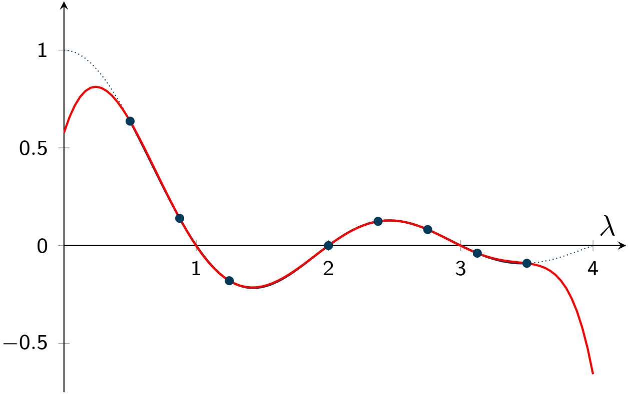
$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_6 n_6(\lambda)$$



Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_7 n_7(\lambda)$$



Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

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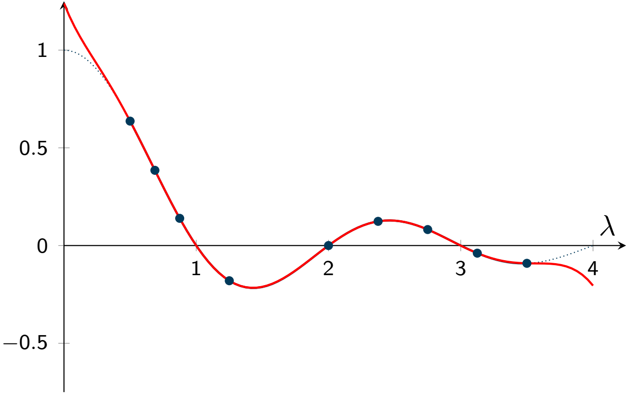
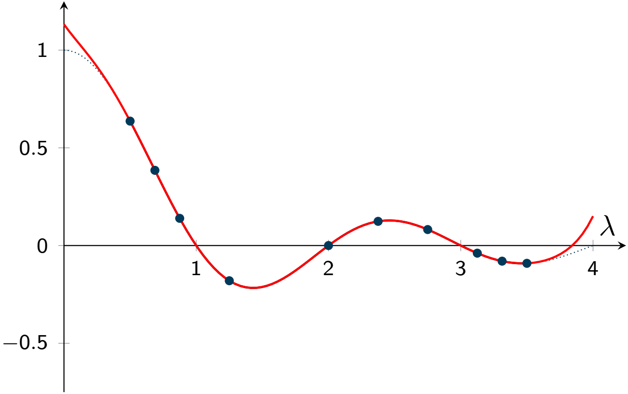


Approximation: Newton interpolation

Dynamic polynomial interpolation (Newton)

$$A(\lambda) \approx A_0 n_0(\lambda) + A_1 n_1(\lambda) + \cdots + A_9 n_9(\lambda)$$

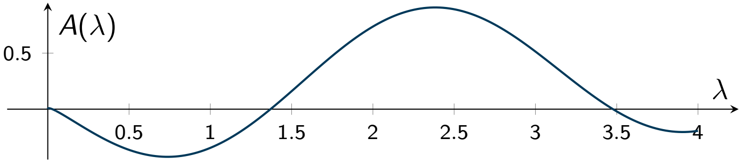


Approximation: Polynomial versus Rational

Scalar nonlinear function:

$$A(\lambda) = 0.2\sqrt{\lambda} - 0.6\sin(2\lambda) = 0$$

with target set: $\Sigma = [0.01, 4]$



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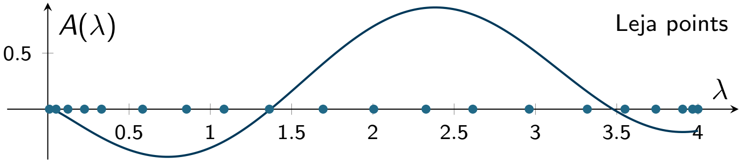

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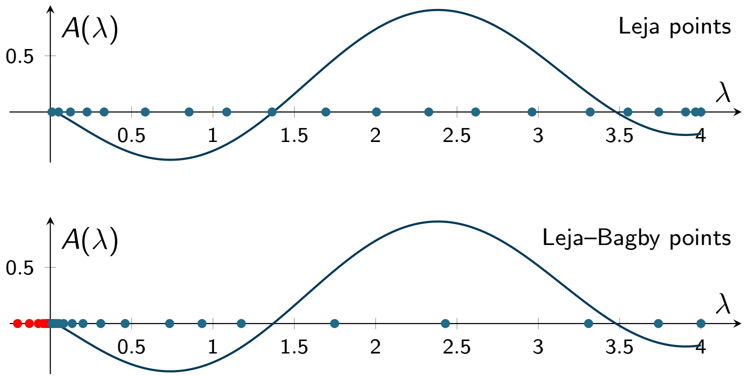

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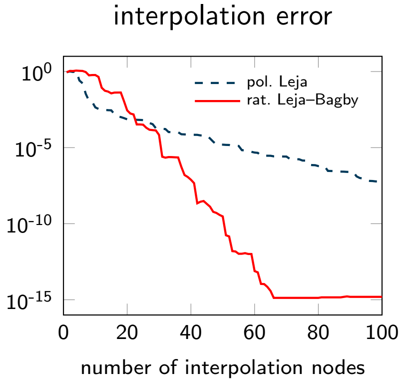
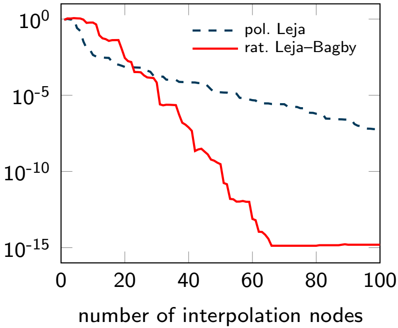


Approximation: Polynomial versus Rational

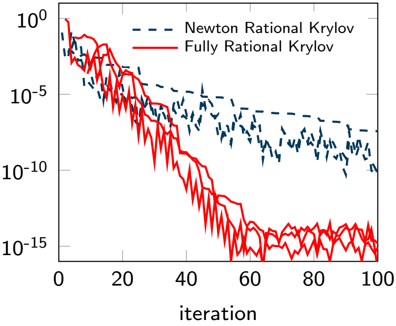
Scalar nonlinear function:

$$A(\lambda) = 0.2\sqrt{\lambda} - 0.6\sin(2\lambda) = 0$$

interpolation error



convergence of eigenvalues



Linearization pencils

NLEP

$A(\lambda)x = 0$

$\Downarrow$

PEP

$P_d(\lambda)x = 0$

$\Downarrow$

GEP

$\mathbf{L}(\lambda)\mathbf{x} = 0$

$\Downarrow$

Solution

Step 2: Linearization

$$P_d(\lambda)x = 0$$

$$\Downarrow$$

$$\mathbf{L}(\lambda)\mathbf{x} = (\mathbf{A} - \lambda\mathbf{B})\mathbf{x} = 0$$

Linearization pencils

Linearization: idea

Second order ODE

$$M\ddot{q} + C\dot{q} + Kq = 0$$

System of first order ODEs

$$\begin{cases} \dot{q}_1 = q_2 \\ M\dot{q}_2 = -Cq_2 - Kq_1 \end{cases}$$

$$\begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

Linearization pencils

Linearization: idea

Second order ODE

$$M\ddot{q} + C\dot{q} + Kq = 0$$

Quadratic eigenvalue problem

$$\left(M\lambda^2 + C\lambda + K\right) x = 0$$

System of first order ODEs

$$\begin{cases} \dot{q}_1 = q_2 \\ M\dot{q}_2 = -Cq_2 - Kq_1 \end{cases}$$

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Linear eigenvalue problem

$$\lambda \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -K & -C \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Linearization pencils

Step 2: Companion linearization

PEP

$P_d(\lambda)x = 0$

$\Rightarrow$

GEP

$\mathbf{L}(\lambda)\mathbf{x} = (\mathbf{A} - \lambda\mathbf{B})\mathbf{x} = 0$

$$P_d(\lambda)x = \left(A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_d\lambda^d\right) x = 0$$

$$\underbrace{\begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_{d-1} \\ & I & & & \\ & & I & & \\ & & & \ddots & \\ & & & & I \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{bmatrix}}_{\mathbf{x}} = \lambda \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & -A_d \\ I & 0 & & & \\ & \ddots & \ddots & & \\ & & I & 0 & \\ & & & I & 0 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{bmatrix}}_{\mathbf{x}}$$

Linearization pencils: Uniform representation

$$\mathbf{L}(\lambda) = \underbrace{\left[\begin{array}{cccc} A_0 & A_1 & \cdots & A_{d-1} \\ \hline \multicolumn{4}{c}{M \otimes I_n} \end{array}\right]}_{\mathbf{A}} - \lambda \underbrace{\left[\begin{array}{cccc} B_0 & B_1 & \cdots & B_{d-1} \\ \hline \multicolumn{4}{c}{N \otimes I_n} \end{array}\right]}_{\mathbf{B}}$$

- Monomial basis [Mackey, Mackey, Mehl, Mehrmann 2006]
- Chebyshev basis [Effenberger, Kressner 2012]
- Lagrange basis [VB, Michiels, Meerbergen 2015]
- Newton/Hermite basis [Amiraslani, Corless, Lancaster 2009]
- Rational monomial basis [Nakatsukasa, Tisseur 2014]
- Rational Newton basis [Güttel, VB, Meerbergen, Michiels 2014]
- Spectral discretization [Jarlebring, Meerbergen, Michiels 2010]
- ...

Solving linear eigenvalue problem

NLEP

$A(\lambda)x = 0$

$\Downarrow$

PEP

$P_d(\lambda)x = 0$

$\Downarrow$

GEP

$\mathbf{L}(\lambda)\mathbf{x} = 0$

$\Downarrow$

Solution

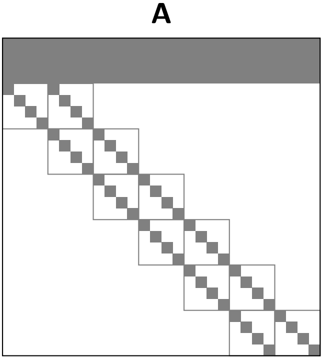
Step 3: Solve generalized linear eigenvalue problem

$$\mathbf{L}(\lambda)\mathbf{x} = (\mathbf{A} - \lambda\mathbf{B})\mathbf{x} = 0$$

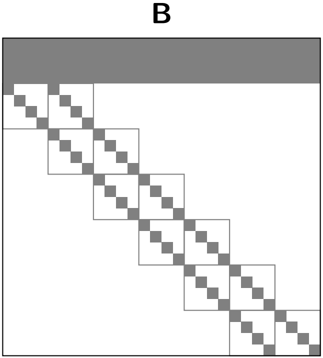
by the rational Krylov method.

Compact Rational Krylov (CORK) framework

$\mathbf{L}(\lambda) =$



$-\lambda$



- full exploitation of structure $\rightsquigarrow$ compact Krylov subspace
- reduction in memory cost
- reduction in computation cost

Compact Rational Krylov (CORK) framework

CORK

Generic class of numerical methods with a lot flexibility:

- polynomial or rational approximation in different bases
- interpolation nodes (and poles)
- shifts in the rational Krylov method
- dynamic / static / hybrid

Compact Rational Krylov variants:

1. Newton Rational Krylov (NRK) method [VB, Meerbergen, Michiels 2013]
2. Fully Rational Krylov (FRK) method [Güttel, VB, Meerbergen, Michiels 2014]
3. Infinite Arnoldi (InfA) method [Jarlebring, Michiels, Meerbergen 2012]
4. . . .

Gun problem

Maxwell's equations + nonlinear waveguide boundary conditions:

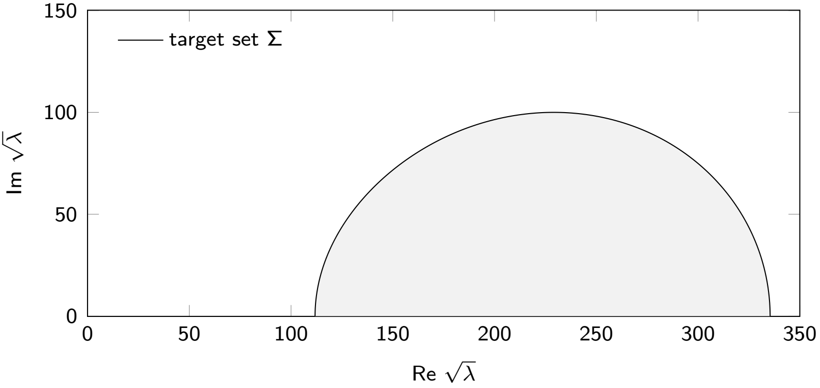
$$A(\lambda)x = \left(K - \lambda M + i\sqrt{\lambda - \alpha_1^2} W_1 + i\sqrt{\lambda - \alpha_2^2} W_2 \right) x = 0,$$

where

- $K, M, W_1, W_2 \in \mathbb{R}^{9956 \times 9956}$
- $\lambda = \omega^2/c^2$
  - ω the wavenumber
  - c the speed of light
- α_1 and α_2 are the cutoff values

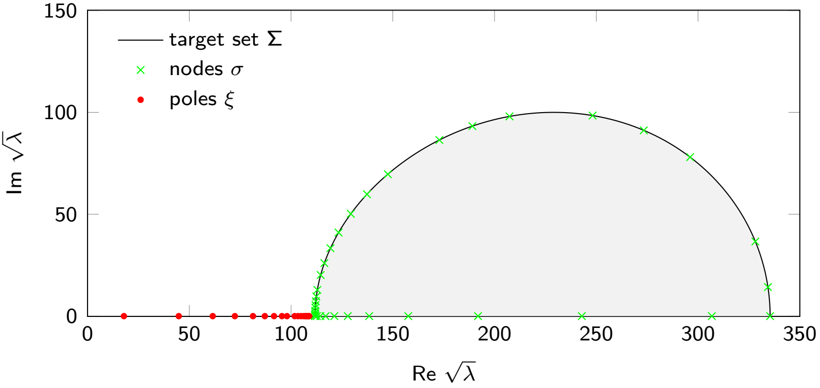
= 'Gun' problem from the NLEVP collection

Gun problem



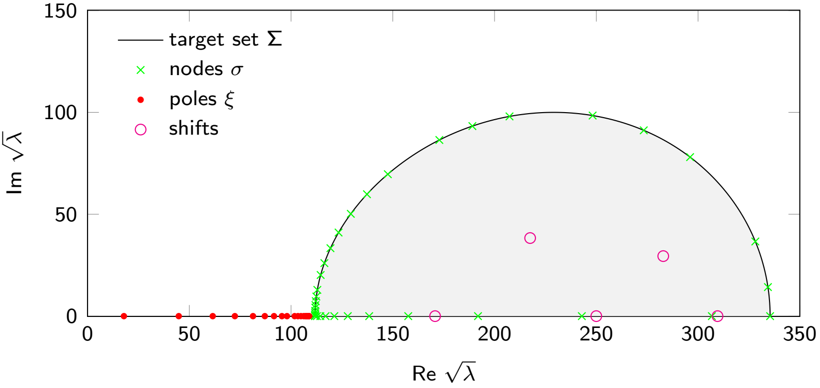
target set Σ
150
100
50
0
Im $\sqrt{\lambda}$
0
50
100
150
200
250
300
350
Re $\sqrt{\lambda}$

Gun problem



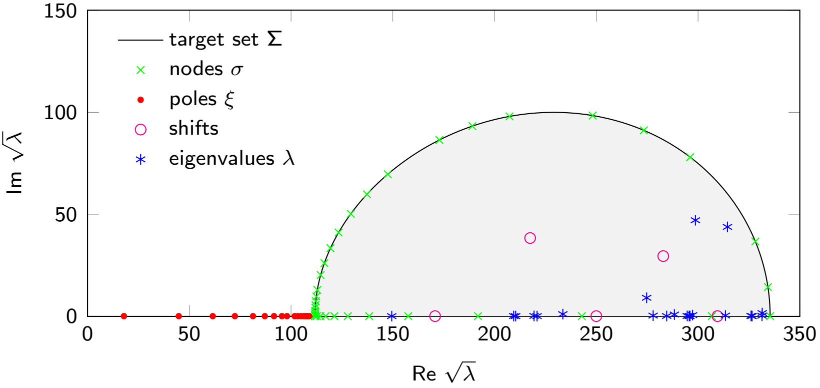
$A(\lambda)$ approximated by a rational Newton polynomial of degree $d = 35$

Gun problem



$A(\lambda)$ approximated by a rational Newton polynomial of degree $d = 35$

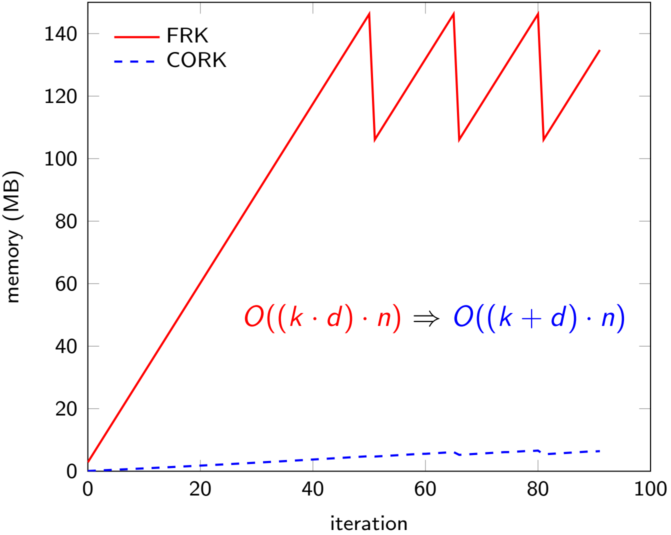
Gun problem



$A(\lambda)$ approximated by a rational Newton polynomial of degree $d = 35$

Compact Rational Krylov (CORK) framework

$n = 10.000 \qquad d = 35 \qquad k = 50$



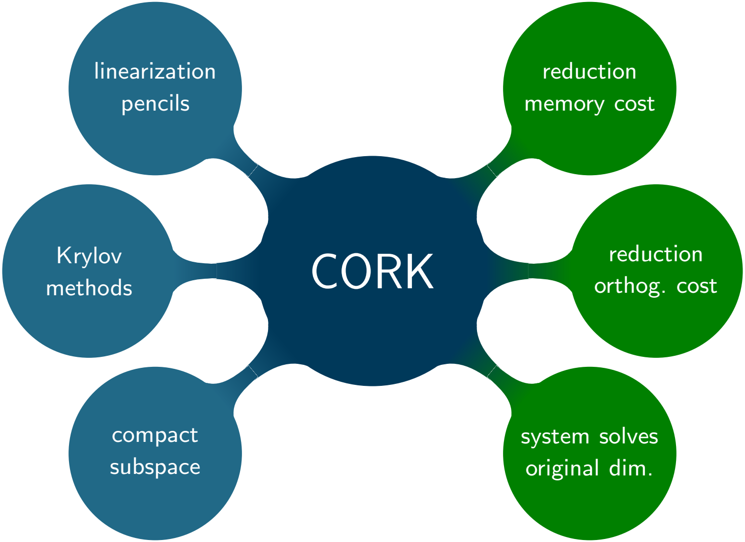
Conclusions

How to solve NLEP's

$$A(\lambda)x = 0$$

1. Approximation via interpolation
2. Linearization
3. Solve linear eigenvalue problem

Compact Rational Krylov (CORK) framework



References

[1] VB, Meerbergen, and Michiels. *Compact rational Krylov methods for nonlinear eigenvalue problems*
SIAM Journal on Matrix Analysis and Applications, 36(2), 820–838, 2015

[2] Güttel, VB, Meerbergen, and Michiels. *NLEIGS: A class of fully rational Krylov methods for nonlinear eigenvalue problems*
SIAM Journal on Scientific Computing, 36 (6), A2842–A2864, 2014

[3] VB, Meerbergen, and Michiels. *A rational Krylov method based on Hermite interpolation for nonlinear eigenvalue problems*
SIAM Journal on Scientific Computing, 35 (1), A327-A350, 2013

CORK & NLEIGS Matlab toolboxes available on my homepage:
`http://www.roelvanbeeumen.be`

References

[1] VB, Meerbergen, and Michiels. *Compact rational Krylov methods for nonlinear eigenvalue problems*
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Thank you for your attention !